Economics 312 **Spring 2013**
Daily Problem #26 **April 1**

Note: Appendices 9B and 9C of HGL discusses these questions.

One of the most common time-series processes is the first-order autoregressive process: AR(1). Suppose that the error term of a time-series regression follows an AR(1) process: $(1-\rho L)e_t = v_t$, or $e_t = \rho e_{t-1} + v_t$, where –1 < ρ < 1 and $v_t \sim N(0,\sigma_v^2)$ is serially uncorrelated white noise.

1. Using the result of Daily Problem #24, show that we can write $e_t = \sum_{s=0}^{\infty} \rho^s v_{t-s}$.

2. Using the property that $\text{cov}(v_t, v_{t-s}) = 0$ for all $s \neq 0$, use the expression in question 1 to show that $\text{var}(e_t) \equiv \sigma_e^2 = \sigma_v^2 \sum_{s=0}^{\infty} (\rho^2)^s = \frac{\sigma_v^2}{1-\rho^2}$. (Hint: Use the equation $e_t = \rho e_{t-1} + v_t$, take the variance of both sides, and note that $\text{var}(e_t) = \sigma_e^2$ for all *t* and that v_t is uncorrelated with anything that happened before *t*.)

3. Show that $\text{cov}(e_t, e_{t-1}) = \rho\sigma_e^2$ and that, more generally, $\text{cov}(e_t, e_{t-s}) = \rho^s \sigma_e^2$. (You can proceed most easily in a manner similar to the hint above.)

4. Using these results, show that the covariance matrix of the error vector $\mathbf{e} \equiv \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_T \end{bmatrix}$ is

$$E(\mathbf{ee}') = \sigma_e^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & & \vdots \\ \rho^2 & \rho & 1 & & \rho^2 \\ \vdots & & & \ddots & \rho \\ \rho^{T-1} & \cdots & \rho^2 & \rho & 1 \end{bmatrix}.$$

5. *(Optional bonus question)* Show that if $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{1-\rho^2}} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$, then the regression

$\mathbf{Py} = \mathbf{PX} + \mathbf{Pe}$ corresponds to the GLS estimator of Appendix 9C, and that

$E\left[\mathbf{Pe}(\mathbf{Pe})'\right] = E[\mathbf{Pee'P'}] = \mathbf{P}E(\mathbf{ee'})\mathbf{P'} = \sigma_v^2 \mathbf{I}_T$ so that the transformed GLS model satisfies the standard MR assumptions.