Economics 312 **Spring 2012**
Daily Problem #19 **February 29**

Suppose that we have a regression model $y_i = \beta_1 + \beta_2 x_i + e_i$, or $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$, in which it is known that the variance of the error term is proportional to some known external variable *z*: $\text{var}(e_i) = z_i \sigma^2$.

1. Show that $\text{cov}(\mathbf{e}) = E(\mathbf{ee}') = \sigma^2 \mathbf{W} = \sigma^2 \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_N \end{bmatrix}$.

2. Show that $\begin{bmatrix} \frac{1}{z_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{z_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{z_N} \end{bmatrix} = \mathbf{W}^{-1}$ by showing that multiplying it by $\mathbf{W}$ gives $\mathbf{I}_N$.

3. Let $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{z_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{z_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{z_N}} \end{bmatrix}$. Show that $\mathbf{PP}' = \mathbf{W}^{-1}$.

4. Consider the "transformed" model $\mathbf{y}^* = \mathbf{X}^*\boldsymbol{\beta} + \mathbf{e}^*$, where $\mathbf{y}^* \equiv \mathbf{Py}$, $\mathbf{X}^* \equiv \mathbf{PX}$, and $\mathbf{e}^* \equiv \mathbf{Pe}$. Show that the representative element of **Py** is $\frac{y_i}{\sqrt{z_i}}$, that the representative element of the first column of **PX** is $\frac{1}{\sqrt{z_i}}$, that the representative element of the second column of **PX** is $\frac{x_i}{\sqrt{z_i}}$, and that the representative element of **Pe** is $\frac{e_i}{\sqrt{z_i}}$.

5. Show that $E\left(\mathbf{e}^* \mathbf{e}^{*\prime}\right) = \sigma^2 \mathbf{I}_N$, so that the transformed model satisfies the SR assumptions leading to the Gauss-Markov Theory and the conclusion that OLS is the best, linear, unbiased estimator of the transformed model.

6. Show that the generalized least-squares estimator (in this case, the weighted least-squares estimator) $\hat{\boldsymbol{\beta}}_{GLS} \equiv \left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1} \mathbf{X}^{*\prime}\mathbf{y}^*$ is equal to $\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{y}\right)$.