Ramsey model

Rationale

- Problem with the Solow model: ad-hoc assumption of constant saving rate
- Will conclusions of Solow model be altered if saving is endogenously determined by utility maximization?
  - Yes, but we will learn a lot about consumption/saving behavior and about dynamic analysis by analyzing it.
- Basic setup of Ramsey model was described by Ramsey in 1928.
- Dynamics were developed by Cass and Koopmans in a growth context in 1965.

Basic setup

- Firms
  - Maximize profit
  - Produce *Y*; hire services of *L* and *K* from households who own them
  - $Y = F(K, AL)$ with usual properties
  - $\frac{\dot{A}}{A} = g$ exogenous as in Solow
- Households
  - Maximize utility
  - Rent *L* and *K* to firms inelastically
  - Buy *Y* for consumption (*C*) and saving/investment
  - Live forever — dynastic interpretation
  - Size of each household grows by $\frac{\dot{L}}{L} = n$ each period.
- All markets are assumed to be perfectly competitive with perfect information and perfect foresight
- Only significant decision in the model is households decided when to consume
  - Saving/dissaving (or investing) is the mechanism for *intertemporal substitution*
  - Households maximize lifetime utility subject to lifetime budget constraint
  - We will analyze this decision process in several steps:
    - First, two-period discrete-time model (Diamond model will use)
    - Next, extend to many periods
    - Then, extend to continuous time and infinite lifetimes
  - On the way, we will establish some important implications for consumption theory (from Chapter 7 of Romer, 16 of coursebook)

Intertemporal budget constraint in discrete time

- Two periods
  - Let K_0 be the amount of capital (the only durable asset) that a household owns at the end of period 0 (beginning of period 1)
  - Household can add to K by saving: choosing C < income
  - W_t = wage income in period t
  - r = real interest rate = return on capital per period (annual compounding for now)
  - $$K_1 = (1+r_1)K_0 + W_1 - C_1$$
    $$K_2 = (1+r_2)K_1 + W_2 - C_2 = (1+r_2)\big((1+r_1)K_0 + W_1 - C_1\big) + W_2 - C_2$$
  - $$(1+r_1)(1+r_2)K_0 + (1+r_2)W_1 + W_2 = K_2 + (1+r_2)C_1 + C_2$$
    $$K_0 + \frac{W_1}{1+r_1} + \frac{W_2}{(1+r_1)(1+r_2)} = \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_1)(1+r_2)} + \frac{K_2}{(1+r_1)(1+r_2)}$$
  - Initial wealth + PV of labor income = PV of consumption + PV of terminal wealth
  - If household leaves no bequest, then last term is zero
  - If, in addition, rate of return is constant, then $r_t = r$, and
    $$K_0 + \frac{W_1}{1+r} + \frac{W_2}{(1+r)^2} = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2}$$
    - Left-hand side is exogenous PV of lifetime wealth (non-human + human capital)
    - Right-hand side poses decision for household: how much to consume in period one vs. period two?
  - Graphing the two-period budget constraint
    - Let $Q_2 = (1+r)^2 K_0 + (1+r)W_1 + W_2$ be $(1 + r)^2$ times the left-hand side of budget constraint (equals lifetime wealth in present value as of period 2), then $C_2 = Q_2 - (1+r)C_1$ is the budget constraint relating consumption in the two periods.
    - The budget constraint is a straight line with slope $-(1+r)$ and vertical intercept Q_2
- Extending to n periods
  - $$K_0 + \sum_{t=1}^{n} \frac{W_t}{\prod_{\tau=1}^{t}(1+r_\tau)} = \sum_{t=1}^{n} \frac{C_t}{\prod_{\tau=1}^{t}(1+r_\tau)} + \frac{K_n}{\prod_{\tau=1}^{n}(1+r_\tau)}$$
- If we assume that terminal wealth is zero, the last term disappears

- In infinite time, we need to assume that the household's capital does not grow at a rate faster than the interest rate, so the limit of the last term is zero:

$$K_0 + \sum_{t=1}^{\infty} \frac{W_t}{\prod_{\tau=1}^{t}(1+r_\tau)} = \sum_{t=1}^{\infty} \frac{C_t}{\prod_{\tau=1}^{t}(1+r_\tau)} + \lim_{n\to\infty} \frac{K_n}{\prod_{\tau=1}^{n}(1+r_\tau)}$$

$$K_0 + \sum_{t=1}^{\infty} \frac{W_t}{\prod_{\tau=1}^{t}(1+r_\tau)} = \sum_{t=1}^{\infty} \frac{C_t}{\prod_{\tau=1}^{t}(1+r_\tau)}$$

- With constant rate of return this becomes

$$K_0 + \sum_{t=1}^{\infty} \frac{W_t}{(1+r)^t} = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$$

Lifetime utility

- Instantaneous utility and lifetime utility
- Two periods: $U = u(C_1) + \frac{1}{1+\rho} u(C_2)$
  - o ρ = marginal rate of time preference (internal discount rate), measures household's impatience
  - o $\rho = 0$ means household values consumption next period as much as this period
  - o $\rho >> 0$ means household is very impatient and discounts future utility heavily
- Extending to n periods or infinite horizon
  - o $U = \sum_{t=1}^{n} \frac{1}{(1+\rho)^t} u(C_t)$
  - o $U = \sum_{t=1}^{\infty} \frac{1}{(1+\rho)^t} u(C_t)$
- Nature of the "felicity" function (instantaneous utility function) u:
  - o $MU = u'(C_t) > 0$
  - o $\frac{dMU}{dC_t} = u''(C_t) < 0$
  - o (Positive but diminishing marginal utility of consumption)
  - o Note convex shape on graph of u vs. C
- Possible functional forms that have appropriate derivatives:
  - o Linear doesn't work because $u''(C) = 0$
  - o Quadratic can force $u''(C) < 0$ but does not have $u'(C) > 0$ for all C

- Most convenient form turns out to be Constant Rate of Risk Aversion form: $u(C) = \frac{C^{1-\theta}}{1-\theta}, \theta > 0.$
  - For this function, $u'(C) = C^{-\theta} > 0$ and $u''(C) = -\theta C^{-\theta-1} < 0$
  - θ is rate of risk aversion that governs how sharply the utility function bends
  - $\theta = 0$ would be linear function
  - $\theta \to \infty$ would have kink
  - $\theta = 1$ is special case in which formula converges to $u(C) = \ln C$.

- Utility function and consumption smoothing
  - Suppose that you are considering how to consume Q of wealth (ignore interest and discounting)
  - Show ½Q in each period and compare to ½$Q \pm X$ in two periods.
  - Average utility is lower with ½$Q \pm X$ in each period than with ½Q in each.
  - Thus, households with convex utility functions prefer smooth planned consumption over lumpy consumption
  - However, high (or low) interest rate might tempt them to consume more in future (present)
- Indifference curves for the two-period utility function
  - Equation for indifference curve: $\bar{U} = u(C_1) + \frac{1}{1+\rho} u(C_2)$
  - To get slope, differentiate equation totally with $d\bar{U} = 0$: $d\bar{U} = 0 = u'(C_1)dC_1 + \frac{1}{1+\rho} u'(C_2)dC_2$. Solve for $\left.\frac{dC_2}{dC_1}\right|_{dU=0} = -(1+\rho)\frac{u'(C_1)}{u'(C_2)}$.
  - Because $u'' < 0$, the indifference curves are concave from above as usual.
  - Along the 45-degree line from the origin, $C_1 = C_2$, so $u'(C_1) = u'(C_2)$ and $\frac{u'(C_1)}{u'(C_2)} = 1$, which means that the slope of the indifference curves along that ray is – (1 + ρ).

Equilibrium in the two-period model

- We can graph the indifference map along with the two-period budget constraint and locate the equilibrium at the tangency.
- Suppose that $r = \rho$
  - Slope of budget constraint is – (1 + r) throughout
  - Slope of indifference curve is – (1 + ρ) at $C_1 = C_2$

  - Thus, if $r = \rho$, then the tangency will occur at $C_1 = C_2$ and the household will choose a flat consumption path: equal consumption in both periods
- Suppose that $r > \rho$
  - In this case, the budget constraint is steeper than the indifference curve at $C_1 = C_2$ and the tangency must be above $C_1 = C_2$.
  - The household chooses higher consumption in the future than in the present
  - The reward to saving (r) exceeds the household's marginal disutility of postponing consumption (ρ), so it chooses an upward-sloping time path for consumption.
- Suppose that $r < \rho$
  - Budget constraint is flatter than the indifference curve at $C_1 = C_2$
  - Tangency must be below $C_1 = C_2$
  - Household chooses higher consumption now and lower in the future: a downward-sloping consumption time path
  - The reward to saving falls short of the household's marginal disutility of postponing consumption, so it consumes more now and less in the future.
- Effect of θ
  - θ governs the amount of curvature in indifference curves
    - High $\theta \Rightarrow$ sharp bend $\Rightarrow$ little effect of $r - \rho$ on consumption path
    - Low $\theta \Rightarrow$ nearly linear IC $\Rightarrow$ strong effect
- Conclusion:

  $r > \rho \Rightarrow C_2 > C_1$

  $r = \rho \Rightarrow C_2 = C_1$

  $r < \rho \Rightarrow C_2 < C_1$

  This is important and quite general result for consumption theory.

Implications of the two-period model for consumption behavior

- Consumption path depends on two things:
  - Present value of lifetime wealth (including future earnings)
    - Determines the height of the consumption path
  - Interest rate in relation to marginal rate of time preference
    - Determines whether path is upward or downward sloping
- Effects of temporary vs. permanent change in income
  - Temporary will have small effect on lifetime wealth
  - Permanent will have large effect
  - Permanent will have MPC near 1, temporary will have MPC near $1/T$, where T is remaining years of life.
- Effects of anticipated vs. unanticipated change in income

- If correctly anticipated, then it is already in the period $t-1$ planned consumption path and there will be no effect on the path or on consumption in the year of the change.
- If unanticipated, then entire path will be revised when information about the change becomes available.
  - If unanticipated change is permanent, then large change in consumption path
  - If temporary, then small change
- Only new information at time t will cause consumption at t to differ from the level projected at time $t-1$.
  - This is the basis of the Hall consumption paper that you will read in a couple of weeks.

Continuous-time consumption decision in growth model

- **Budget constraint**
  - Recall infinite-horizon budget constraint (with limiting condition on terminal wealth): $K_0 + \sum_{t=1}^{\infty} \frac{W_t}{\prod_{\tau=1}^{t}(1+r_\tau)} = \sum_{t=1}^{\infty} \frac{C_t}{\prod_{\tau=1}^{t}(1+r_\tau)}$ or $K_0 + \sum_{t=1}^{\infty} \frac{W_t}{(1+r)^t} = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$ if r is constant
  - When we convert to continuous time, we change
    - Notations from C_t to $C(t)$
    - Summations to integrals (starting at $t = 0$)
    - From annual compounding to continuous (instantaneous) compounding
      - $\frac{1}{(1+r)^t} \rightarrow e^{-rt}$
      - $\frac{1}{\prod_{\tau=1}^{t}(1+r_\tau)} \rightarrow e^{-\int_0^t r(\tau)d\tau} \equiv e^{-R(t)}$
  - Infinite-horizon, continuous-time budget constraint for an individual person looking forward from time 0 is $\frac{K(0)}{L(0)} + \int_0^\infty e^{-R(t)} W(t)\,dt = \int_0^\infty e^{-R(t)} C(t)\,dt$, where W is the wage of one worker per period and C is consumption of one worker per period. $\frac{K(0)}{L(0)}$ is the amount of capital owned by one worker at time 0. (Remember that $R(t) = rt$ if the return on capital is constant, which makes the discount factor more familiar.)

- **Family size**: There are H households in the economy with $\frac{L(t)}{H}$ individuals in each household at time t.
  - Note that we assume that population growth occurs through increases in household size (reproduction), not through new households entering (immigration). This is important because it means we can assume that existing people care about their children in a way that they probably wouldn't care about unrelated immigrants.
  - The budget constraint at the household level is $\frac{K(0)}{H}+\int_0^\infty e^{-R(t)}W(t)\frac{L(t)}{H}dt=\int_0^\infty e^{-R(t)}C(t)\frac{L(t)}{H}dt$, where we have used capital per household rather than capital per worker and augmented the earnings and consumption values into per-household measures by multiplying by the number of workers per household.
- **Translating into per-effective-worker units**. As in the Solow model, our steady-state equilibrium will have growing levels of per-person variables like W and C, but stable levels of the corresponding per-effective-worker variables: $w(t)\equiv\frac{W(t)}{A(t)}=$ earnings per efficiency unit of labor and $c(t)\equiv\frac{C(t)}{A(t)}=$ consumption per efficiency unit of labor.
  - Note that
    - C = Total consumption / # of workers = $CONS/L$
    - c = Total consumption / # of effective workers = $CONS/AL$
  - Using definitions above, $W(t)=w(t)A(t)$ and $C(t)=c(t)A(t)$, so $\frac{K(0)}{H}+\int_0^\infty e^{-R(t)}w(t)\frac{A(t)L(t)}{H}dt=\int_0^\infty e^{-R(t)}c(t)\frac{A(t)L(t)}{H}dt$
  - According to the equations of motion of technology and the labor force
    - $\frac{\dot{A}(t)}{A(t)}=g\Rightarrow A(t)=A(0)e^{gt}$
    - $\frac{\dot{L}(t)}{L(t)}=n\Rightarrow L(t)=L(0)e^{nt}$
  - If we define $k(t)\equiv\frac{K(t)}{A(t)L(t)}$ as in the Solow model, then $K(0)=A(0)L(0)k(0)$.

- Thus,

$$k(0)\frac{A(0)L(0)}{H}+\int_0^\infty e^{-R(t)}w(t)e^{gt}e^{nt}\frac{A(0)L(0)}{H}dt=\int_0^\infty e^{-R(t)}c(t)e^{gt}e^{nt}\frac{A(0)L(0)}{H}dt$$

- We can cancel out the AL/H term to get our final intertemporal budget constraint: $k(0)+\int_0^\infty e^{-R(t)}w(t)e^{(g+n)t}dt=\int_0^\infty e^{-R(t)}c(t)e^{(g+n)t}dt$
- Note that the wage per efficiency unit is $\frac{\partial Y(t)}{\partial\left[A(t)L(t)\right]}=f\left(k(t)\right)-k(t)f'\left(k(t)\right)$, so the budget constraint depends on the evolution of two variables over time: *k* and *c*. These will be the central variables of our growth-model analysis.

Dynamic utility in continuous time

- Recall utility function from discrete-time model: $U=\sum_{t=1}^{\infty}\frac{1}{(1+\rho)^t}u(C_t)$
- In continuous time, the utility function changes mirror those of budget constraint
  - Notations from C_t to $C(t)$
  - Summations to integrals (starting at $t = 0$)
  - From annual compounding to continuous (instantaneous) compounding
- Continuous-time utility function at an individual level is $U=\int_0^\infty e^{-\rho t}u\left(C(t)\right)dt$
- Following the same step as with the budget constraint, we first convert to the household level by multiplying utility by the number of people per household:

$$U=\int_0^\infty e^{-\rho t}u\left(C(t)\right)\frac{L(t)}{H}dt\ .$$

- Plugging in the CRRA utility function

$$u\left(C(t)\right)=\frac{C(t)^{1-\theta}}{1-\theta}=\frac{\left[A(t)c(t)\right]^{1-\theta}}{1-\theta}=\left[A(0)e^{gt}\right]^{1-\theta}\frac{c(t)^{1-\theta}}{1-\theta},$$

$$\begin{aligned}U&=\int_0^\infty e^{-\rho t}\left[A(0)e^{gt}\right]^{1-\theta}\frac{c(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt\\&=\int_0^\infty e^{-\rho t}\left[A(0)e^{gt}\right]^{1-\theta}\frac{c(t)^{1-\theta}}{1-\theta}\frac{L(0)e^{nt}}{H}dt\\&=\frac{A(0)^{1-\theta}L(0)}{H}\int_0^\infty e^{-(\rho-n-(1-\theta)g)t}\frac{c(t)^{1-\theta}}{1-\theta}dt\\&\equiv B\int_0^\infty e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt\end{aligned}$$

Intertemporal utility maximization

- Formal mathematical maximization problem:

$$\max_{c(t)} \left\{ B\int_0^\infty e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \right\},$$

subject to $k(0)+\int_0^\infty e^{-R(t)} w(t) e^{(g+n)t} dt = \int_0^\infty e^{-R(t)} c(t) e^{(g+n)t} dt$

and $\lim_{t\to\infty}\left[e^{-R(t)} e^{(n+g)t} k(t) \right] \geq 0.$

  - The last constraint is the "no Ponzi scheme" constraint that prevents households from driving their wealth infinitely negative as time passes.
  - No one would lend to a household that did this, so they wouldn't be able to do it.
- This is a problem in the calculus of variations. We often call this kind of problem "dynamic control theory" in economics.
- We won't explore the solution method in detail. (Romer sketches the solution on page 54.
- The solution for the optimal consumption path consists of two parts
  - The **consumption Euler equation** $\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho-\theta g}{\theta}$, which describes the growth rate of *c* at each point along the consumption path.
    - The Euler equation tells the "slope" (in growth rate terms) of the path at each point, but doesn't determine the level.
  - The **budget constraint** $k(0)+\int_0^\infty e^{-R(t)} w(t) e^{(g+n)t} dt = \int_0^\infty e^{-R(t)} c(t) e^{(g+n)t} dt$ that determines which of the infinite number of parallel consumption paths the household can afford. (It chooses the highest one it can afford.)
  - Romer shows in footnote 9 on page 56 that the initial (time 0) value on the consumption path is

$$c(0) = \frac{k(0)+\int_0^\infty e^{-R(t)} w(t) e^{(g+n)t} dt}{\int_0^\infty e^{\frac{(1-\theta)R(t)+(\theta n-\rho)t}{\theta}} dt}$$

$$= \frac{k(0)+\int_0^\infty e^{-R(t)} \left[f(k(t)) - k(t) f'(k(t)) \right] e^{(g+n)t} dt}{\int_0^\infty e^{\frac{(1-\theta)R(t)+(\theta n-\rho)t}{\theta}} dt}.$$

    Note that *R*(*t*) depends on $r(\tau)$ at all points in time between 0 and *t*, and that $r(\tau) = f'(k(\tau)) - \delta$ is the net return on capital at time τ.

    Thus, *c*(0) depends only on the future time path of *k* (and the parameters of the model).
- **Intuition of the Euler equation**

- Note that c is consumption per effective worker.
  - To get this back to consumption per worker, we use $C = cA$.
  - This means $\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(t)}{c(t)} + g = \frac{r(t) - \rho - \theta g}{\theta} + g = \frac{r(t) - \rho}{\theta}$.
- $\dot{C} > 0$ means individuals are choosing a rising consumption path at moment t, so their consumption shortly after t is higher than at t.
- Correspondingly, $\dot{C} < 0$ means that an individual's consumption is lower at a moment after t than at t, and
- $\dot{C} = 0$ means that consumption per person is the same just after t as at t.
- From the equation:
  - $r(t) > \rho \Rightarrow \dot{C} > 0$
    $r(t) = \rho \Rightarrow \dot{C} = 0$
    $r(t) < \rho \Rightarrow \dot{C} < 0$
- These conditions correspond exactly to those of the two-period model:
  - $r > \rho \Rightarrow C_2 > C_1$
    $r = \rho \Rightarrow C_2 = C_1$
    $r < \rho \Rightarrow C_2 < C_1$
- The intuition is the same: people will choose a rising, flat, or falling consumption path per person (at moment *t*) depending on whether the reward to saving (return to capital = interest rate) exceeds, equals, or falls short of their marginal rate of time preference.
- **How much does a change in *r* affect the consumption decision?**
  - Change in *r* is change in slope of budget constraint
  - How far this changes the optimal consumption point depends on amount of curvature in indifference curves
  - Large θ ⇒ lots of curvature; θ → 0 implies straight line
    - Large θ in denominator means given gap between *r* and ρ leads to small change in consumption path
    - Large θ means that households do not like to substitute consumption over time: want to stick to smooth consumption path regardless of *r*
    - 1/θ is the elasticity of intertemporal substitution
  - Small θ ⇒ nearly linear indifference curves
    - Small θ in denominator means large change in consumption resulting from given gap between *r* and ρ
    - Small θ means households are OK with substituting over time

Aggregate dynamics of Ramsey growth model