Basic properties of the generalized Leontief cost function

(This is based on Section 9.2 of Berndt's text.)

The generalized Leontief (GL) cost function has the form

$$C = Y \cdot \left[\sum_{i=1}^{k} \sum_{j=1}^{k} d_{ij} \left(P_i P_j \right)^{1/2} \right],$$

Where C is total cost, Y is total output, k is the number of inputs, P_i is the price of the ith input, and the d parameters are coefficients that satisfy the normalization restriction $d_{ij} = d_{ji}$.

According to Shephard's Lemma, the cost-minimizing demand for each input i is equal to the partial derivative of cost with respect to P_i:

$$X_i = \frac{\partial C}{\partial P_i} = Y \cdot \left[2 \sum_{j=1}^{k} \frac{1}{2} d_{ij} P_j^{1/2} P_i^{-1/2} \right] = Y \cdot \sum_{j=1}^{k} d_{ij} \left(\frac{P_j}{P_i} \right)^{1/2}.$$

The 2 in front of the single summation in the middle expression is present because each i,j pair from which $i \neq j$ occurs twice. In the term $i = j$, the two square roots become P_i, so there is no ½ out in front.

If we divide the input-demand equation above by total output, we get an input-output ratio that is a linear function of the square-roots of the relative price terms:

$$\frac{X_i}{Y} = \sum_{j=1}^{k} d_{ij} \left(\frac{P_j}{P_i} \right)^{1/2} = d_{ii} + \sum_{j \neq i} d_{ij} \left(\frac{P_j}{P_i} \right)^{1/2},$$

with the last inequality following because $\left(\frac{P_i}{P_i} \right)^{1/2} = 1$.

Research on production technology usually emphasized a four-input system with labor, capital, energy, and materials as the inputs. In this system, there are four related input-demand functions:

$$\frac{K}{Y} = d_{KK} + d_{KL} \left(\frac{P_L}{P_K} \right)^{1/2} + d_{KE} \left(\frac{P_E}{P_K} \right)^{1/2} + d_{KM} \left(\frac{P_M}{P_K} \right)^{1/2}$$

$$\frac{L}{Y} = d_{LL} + d_{KL} \left(\frac{P_K}{P_L} \right)^{1/2} + d_{LE} \left(\frac{P_E}{P_L} \right)^{1/2} + d_{LM} \left(\frac{P_M}{P_L} \right)^{1/2}$$

$$\frac{E}{Y} = d_{EE} + d_{KE} \left(\frac{P_K}{P_E} \right)^{1/2} + d_{LE} \left(\frac{P_L}{P_E} \right)^{1/2} + d_{EM} \left(\frac{P_M}{P_E} \right)^{1/2}$$

$$\frac{M}{Y} = d_{MM} + d_{KM} \left(\frac{P_K}{P_M} \right)^{1/2} + d_{LM} \left(\frac{P_L}{P_M} \right)^{1/2} + d_{EM} \left(\frac{P_E}{P_M} \right)^{1/2}$$

In order to estimate these equations, we add a linear error term to the end of each. Because of the normalization that $d_{ij} = d_{ji}$, these equations have common parameters. For example, d_{KE} appears in both the capital-demand equation and the energy-demand equation. These “symmetry restrictions” can be tested if the input-demand equations are estimated as a system. Moreover, it is highly likely that the error terms across equations will be correlated, so system estimation is likely to be more efficient as well.