Week 2: Growth Models with Optimal Saving

Introduction

- How will households choose optimal pattern of saving and consumption?
  - Need to specify an intertemporal utility function describing the relative value of consuming at various points in time
  - Based on utility-maximization problem, characterize consumption/saving behavior and replace constant-saving-rate assumption with optimal saving rule
  - Describe steady-state equilibrium and path of convergence to steady-state path
- Finite vs. infinite lifetimes?
  - Infinite lifetimes are unrealistic, unless we think of family dynasties in which parents care about their children's future utility.
  - Finite lifetimes allow for life-cycle effects that cannot be modeled with infinitely lived agents who never retire or die.
- Two modeling frameworks that are both commonly used in modern growth analysis:
  - **Ramsey model** (as developed by Cass and Koopmans in the 1960s based on Ramsey's framework from 1928) is in continuous time and has infinite lifetimes.
  - **Diamond model** is in discrete time and has a two-period life cycle for each agent, with overlapping generations (OLG) and no linkages across generations.
- As with most (good) models, each has its strengths and weaknesses.
  - Some questions are more easily answered and modeled by the Ramsey framework; others by the OLG model.

Framework of Ramsey Model

Rationale

- Problem with the Solow model: ad-hoc assumption of constant saving rate
- Will conclusions of Solow model be altered if saving is endogenously determined by utility maximization?
  - Yes, but we will learn a lot about consumption/saving behavior and about dynamic analysis by analyzing it.
- Basic setup of Ramsey model was described by Ramsey in 1928.
- Dynamics were developed by Cass and Koopmans in a growth context in 1965.

Basic setup

- Firms

- Maximize profit
- Produce Y; hire services of L and K from households who own them
- $Y = F(K, AL)$ with usual properties
- $\frac{\dot{A}}{A} = g$ exogenous as in Solow

- Households
  - Maximize utility
  - Rent L and K to firms inelastically
  - Buy Y for consumption (C) and saving/investment
  - Live forever — dynastic interpretation
  - Size of each household grows by $\frac{\dot{L}}{L} = n$ each period.
- All markets are assumed to be perfectly competitive with perfect information and perfect foresight
- Only significant decision in the model is households decided when to consume
  - Saving/dissaving (or investing) is the mechanism for *intertemporal substitution*
  - Households maximize lifetime utility subject to lifetime budget constraint
  - We will analyze this decision process in several steps:
    - First, two-period discrete-time model (Diamond model will use this setup)
    - Next, extend to many periods
    - Then, extend to continuous time and infinite lifetimes
  - On the way, we will establish some important implications for consumption theory (from Chapter 8 of Romer, 16 of coursebook)

Consumption and Saving

Intertemporal budget constraint in discrete time

- Two periods
  - Let K_0 be the amount of capital (the only durable asset) that a household owns at the end of period 0 (beginning of period 1)
  - Household can add to K by saving: choosing $C <$ income
  - W_t = wage income in period t
  - r = real interest rate = return on capital per period (annual compounding for now)
  - Skip the intermediate steps here:

    $K_1 = (1 + r_1)K_0 + W_1 - C_1$

    $K_2 = (1 + r_2)K_1 + W_2 - C_2 = (1 + r_2)\left((1 + r_1)K_0 + W_1 - C_1\right) + W_2 - C_2$

$$\left(1+r_1\right)\left(1+r_2\right)K_0+\left(1+r_2\right)W_1+W_2=K_2+\left(1+r_2\right)C_1+C_2$$

- o $K_0+\dfrac{W_1}{1+r_1}+\dfrac{W_2}{\left(1+r_1\right)\left(1+r_2\right)}=\dfrac{C_1}{1+r_1}+\dfrac{C_2}{\left(1+r_1\right)\left(1+r_2\right)}+\dfrac{K_2}{\left(1+r_1\right)\left(1+r_2\right)}$
- o Initial wealth + PV of labor income = PV of consumption + PV of terminal wealth
- o If household leaves no bequest, then last term is zero
- o If, in addition, rate of return is constant, then $r_t = r$, and

  $$K_0+\frac{W_1}{1+r}+\frac{W_2}{\left(1+r\right)^2}=\frac{C_1}{1+r}+\frac{C_2}{\left(1+r\right)^2}$$

  - Left-hand side is exogenous PV of lifetime wealth (non-human + human capital)
  - Right-hand side poses decision for household: how much to consume in period one vs. period two?
- o Graphing the two-period budget constraint
  - Let $Q_2=\left(1+r\right)^2K_0+\left(1+r\right)W_1+W_2$ be $(1+r)^2$ times the left-hand side of budget constraint (equals lifetime wealth in present value as of period 2), then $C_2=Q_2-\left(1+r\right)C_1$ is the budget constraint relating consumption in the two periods.
  - The budget constraint is a straight line with slope $-\left(1+r\right)$ and vertical intercept Q_2

- Extending to n periods
  - o $K_0+\displaystyle\sum_{t=1}^{n}\frac{W_t}{\prod_{\tau=1}^{t}\left(1+r_\tau\right)}=\sum_{t=1}^{n}\frac{C_t}{\prod_{\tau=1}^{t}\left(1+r_\tau\right)}+\frac{K_n}{\prod_{\tau=1}^{n}\left(1+r_\tau\right)}$
- If we assume that terminal wealth is zero, the last term disappears
- In infinite time, we need to assume that the household's capital does not grow at a rate faster than the interest rate, so the limit of the last term is zero:

$$K_0+\sum_{t=1}^{\infty}\frac{W_t}{\prod_{\tau=1}^{t}\left(1+r_\tau\right)}=\sum_{t=1}^{\infty}\frac{C_t}{\prod_{\tau=1}^{t}\left(1+r_\tau\right)}+\lim_{n\to\infty}\frac{K_n}{\prod_{\tau=1}^{n}\left(1+r_\tau\right)}$$

$$K_0+\sum_{t=1}^{\infty}\frac{W_t}{\prod_{\tau=1}^{t}\left(1+r_\tau\right)}=\sum_{t=1}^{\infty}\frac{C_t}{\prod_{\tau=1}^{t}\left(1+r_\tau\right)}$$

- Get to here as quickly as possible: With constant rate of return this becomes

$$K_0+\sum_{t=1}^{\infty}\frac{W_t}{\left(1+r\right)^t}=\sum_{t=1}^{\infty}\frac{C_t}{\left(1+r\right)^t}$$

Lifetime utility

- Instantaneous utility and lifetime utility
- Two periods: $U=u\left(C_1\right)+\dfrac{1}{1+\rho}u\left(C_2\right)$

  - ρ = marginal rate of time preference (internal discount rate), measures household's impatience
  - $\rho = 0$ means household values consumption next period as much as this period
  - $\rho >> 0$ means household is very impatient and discounts future utility heavily
- Extending to n periods or infinite horizon
  - $U = \sum_{t=1}^{n} \frac{1}{(1+\rho)^t} u(C_t)$
  - $U = \sum_{t=1}^{\infty} \frac{1}{(1+\rho)^t} u(C_t)$
- Nature of the "felicity" function (instantaneous utility function) u:
  - $MU = u'(C_t) > 0$
  - $\frac{dMU}{dC_t} = u''(C_t) < 0$
  - (Positive but diminishing marginal utility of consumption)
  - Note convex shape on graph of u vs. C
- Possible functional forms that have appropriate derivatives:
  - Linear doesn't work because $u''(C) = 0$
  - Quadratic can force $u''(C) < 0$ but does not have $u'(C) > 0$ for all C
  - Most convenient form turns out to be Constant Relative Risk Aversion form:
    $u(C) = \frac{C^{1-\theta}}{1-\theta}, \theta > 0.$
    - For this function, $u'(C) = C^{-\theta} > 0$ and $u''(C) = -\theta C^{-\theta-1} < 0$
    - θ is rate of risk aversion that governs how sharply the utility function bends
    - $\theta = 0$ would be linear function
    - $\theta \rightarrow \infty$ would have kink
    - $\theta = 1$ is special case in which formula converges to $u(C) = \ln C$.
- Utility function and consumption smoothing
  - Suppose that you are considering how to consume Q of wealth (ignore interest and discounting)
  - Show ½Q in each period and compare to ½$Q \pm X$ in two periods.
  - Average utility is lower with ½$Q \pm X$ in each period than with ½Q in each.
  - Thus, households with convex utility functions prefer smooth planned consumption over lumpy consumption
  - However, high (or low) interest rate might tempt them to consume more in future (present)
- Indifference curves for the two-period utility function

- Equation for indifference curve: $\bar{U} = u(C_1) + \frac{1}{1+\rho} u(C_2)$
- To get slope, differentiate equation totally with $d\bar{U} = 0$:
  $d\bar{U} = 0 = u'(C_1)dC_1 + \frac{1}{1+\rho} u'(C_2)dC_2$. Solve for $\left.\frac{dC_2}{dC_1}\right|_{dU=0} = -(1+\rho)\frac{u'(C_1)}{u'(C_2)}$.
- Because $u'' < 0$, the indifference curves are concave from above as usual.
- Along the 45-degree line from the origin, $C_1 = C_2$, so $u'(C_1) = u'(C_2)$ and $\frac{u'(C_1)}{u'(C_2)} = 1$, which means that the slope of the indifference curves along that ray is $-(1+\rho)$.

Equilibrium in the two-period model

- We can graph the indifference map along with the two-period budget constraint and locate the equilibrium at the tangency.
- Suppose that $r = \rho$
  - Slope of budget constraint is $-(1+r)$ throughout
  - Slope of indifference curve is $-(1+\rho)$ at $C_1 = C_2$
  - Thus, if $r = \rho$, then the tangency will occur at $C_1 = C_2$ and the household will choose a flat consumption path: equal consumption in both periods
- Suppose that $r > \rho$
  - In this case, the budget constraint is steeper than the indifference curve at $C_1 = C_2$ and the tangency must be above $C_1 = C_2$.
  - The household chooses higher consumption in the future than in the present
  - The reward to saving (r) exceeds the household's marginal disutility of postponing consumption (ρ), so it chooses an upward-sloping time path for consumption.
- Suppose that $r < \rho$
  - Budget constraint is flatter than the indifference curve at $C_1 = C_2$
  - Tangency must be below $C_1 = C_2$
  - Household chooses higher consumption now and lower in the future: a downward-sloping consumption time path
  - The reward to saving falls short of the household's marginal disutility of postponing consumption, so it consumes more now and less in the future.
- Effect of θ
  - θ governs the amount of curvature in indifference curves
    - High $\theta \Rightarrow$ sharp bend $\Rightarrow$ little effect of $r - \rho$ on consumption path
    - Low $\theta \Rightarrow$ nearly linear IC $\Rightarrow$ strong effect

- Conclusion:

  $r > \rho \Rightarrow C_2 > C_1$

  $r = \rho \Rightarrow C_2 = C_1$

  $r < \rho \Rightarrow C_2 < C_1$

  This is important and quite general result for consumption theory.

Implications of the two-period model for consumption behavior

- Consumption path depends on two things:
  - Present value of lifetime wealth (including future earnings)
    - Determines the height of the consumption path
  - Interest rate in relation to marginal rate of time preference
    - Determines whether path is upward or downward sloping
- Effects of temporary vs. permanent change in income
  - Temporary will have small effect on lifetime wealth
  - Permanent will have large effect
  - Permanent will have MPC near 1, temporary will have MPC near $1/T$, where T is remaining years of life.
- Effects of anticipated vs. unanticipated change in income
  - If correctly anticipated, then it is already in the period $t-1$ planned consumption path and there will be no effect on the path or on consumption in the year of the change.
  - If unanticipated, then entire path will be revised when information about the change becomes available.
    - If unanticipated change is permanent, then large change in consumption path
    - If temporary, then small change
  - Only new information at time t will cause consumption at t to differ from the level projected at time $t-1$.
    - This is the basis of the Hall consumption paper that you may read in a couple of weeks.

Continuous-time consumption decision in growth model

- **Budget constraint**
  - Recall infinite-horizon budget constraint (with limiting condition on terminal wealth): $K_0 + \sum_{t=1}^{\infty} \frac{W_t}{\prod_{\tau=1}^{t}(1+r_\tau)} = \sum_{t=1}^{\infty} \frac{C_t}{\prod_{\tau=1}^{t}(1+r_\tau)}$ or $K_0 + \sum_{t=1}^{\infty} \frac{W_t}{(1+r)^t} = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$ if r is constant
  - When we convert to continuous time, we change
    - Notations from C_t to $C(t)$
    - Summations to integrals (starting at $t = 0$)

- From annual compounding to continuous (instantaneous) compounding
  - $\dfrac{1}{(1+r)^t} \to e^{-rt}$
  - $\dfrac{1}{\prod_{\tau=1}^{t}(1+r_\tau)} \to e^{-\int_0^t r(\tau)d\tau} \equiv e^{-R(t)}$

- Infinite-horizon, continuous-time budget constraint for an individual person looking forward from time 0 is $\dfrac{K(0)}{L(0)} + \int_0^\infty e^{-R(t)}W(t)\,dt = \int_0^\infty e^{-R(t)}C(t)\,dt$, where W is the wage of one worker per period and C is consumption of one worker per period. $\dfrac{K(0)}{L(0)}$ is the amount of capital owned by one worker at time 0. (Remember that $R(t) = rt$ if the return on capital is constant, which makes the discount factor more familiar.)
- **Family size**: There are H households in the economy with $\dfrac{L(t)}{H}$ individuals in each household at time t.
  - Note that we assume that population growth occurs through increases in household size (reproduction), not through new households entering (immigration). This is important because it means we can assume that existing people care about their children in a way that they probably wouldn't care about unrelated immigrants.
  - The budget constraint at the household level is $\dfrac{K(0)}{H} + \int_0^\infty e^{-R(t)}W(t)\dfrac{L(t)}{H}dt = \int_0^\infty e^{-R(t)}C(t)\dfrac{L(t)}{H}dt$, where we have used capital per household rather than capital per worker and augmented the earnings and consumption values into per-household measures by multiplying by the number of workers per household.
- **Translating into per-effective-worker units**. As in the Solow model, our steady-state equilibrium will have growing levels of per-person variables like W and C, but stable levels of the corresponding per-effective-worker variables:

  $w(t) \equiv \dfrac{W(t)}{A(t)} =$ earnings per efficiency unit of labor and

  $c(t) \equiv \dfrac{C(t)}{A(t)} =$ consumption per efficiency unit of labor.
  - Note that
    - C = Total consumption / # of workers = $CONS/L$
    - c = Total consumption / # of effective workers = $CONS/AL$

- Using definitions above, $W(t) = w(t)A(t)$ and $C(t) = c(t)A(t)$, so
$$\frac{K(0)}{H} + \int_0^\infty e^{-R(t)} w(t) \frac{A(t)L(t)}{H} dt = \int_0^\infty e^{-R(t)} c(t) \frac{A(t)L(t)}{H} dt$$
- According to the equations of motion of technology and the labor force
  - $\frac{\dot{A}(t)}{A(t)} = g \Rightarrow A(t) = A(0)e^{gt}$
  - $\frac{\dot{L}(t)}{L(t)} = n \Rightarrow L(t) = L(0)e^{nt}$
- If we define $k(t) \equiv \frac{K(t)}{A(t)L(t)}$ as in the Solow model, then $K(0) = A(0)L(0)k(0)$.
- Thus,
$$k(0)\frac{A(0)L(0)}{H} + \int_0^\infty e^{-R(t)} w(t) e^{gt} e^{nt} \frac{A(0)L(0)}{H} dt = \int_0^\infty e^{-R(t)} c(t) e^{gt} e^{nt} \frac{A(0)L(0)}{H} dt$$
- We can cancel out the AL/H term to get our final intertemporal budget constraint: $k(0) + \int_0^\infty e^{-R(t)} w(t) e^{(g+n)t} dt = \int_0^\infty e^{-R(t)} c(t) e^{(g+n)t} dt$
- Note that the wage per efficiency unit is $\frac{\partial Y(t)}{\partial\left[A(t)L(t)\right]} = f\left(k(t)\right) - k(t)f'\left(k(t)\right)$, so the budget constraint depends on the evolution of two variables over time: *k* and *c*. These will be the central variables of our growth-model analysis.

Dynamic utility in continuous time

- Recall utility function from discrete-time model: $U = \sum_{t=1}^{\infty} \frac{1}{(1+\rho)^t} u(C_t)$
- In continuous time, the utility function changes mirror those of budget constraint
  - Notations from C_t to $C(t)$
  - Summations to integrals (starting at $t = 0$)
  - From annual compounding to continuous (instantaneous) compounding
- Continuous-time utility function at an individual level is $U = \int_0^\infty e^{-\rho t} u\left(C(t)\right) dt$
- Following the same step as with the budget constraint, we first convert to the household level by multiplying utility by the number of people per household:
$$U = \int_0^\infty e^{-\rho t} u\left(C(t)\right) \frac{L(t)}{H} dt .$$

- Plugging in the CRRA utility function:

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} = \frac{\left[A(t)c(t)\right]^{1-\theta}}{1-\theta} = \left[A(0)e^{gt}\right]^{1-\theta}\frac{c(t)^{1-\theta}}{1-\theta},$$

$$\begin{aligned}
U &= \int_0^\infty e^{-\rho t}\left[A(0)e^{gt}\right]^{1-\theta}\frac{c(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt \\
&= \int_0^\infty e^{-\rho t}\left[A(0)e^{gt}\right]^{1-\theta}\frac{c(t)^{1-\theta}}{1-\theta}\frac{L(0)e^{nt}}{H}dt \\
&= \frac{A(0)^{1-\theta}L(0)}{H}\int_0^\infty e^{-(\rho-n-(1-\theta)g)t}\frac{c(t)^{1-\theta}}{1-\theta}dt \\
&\equiv B\int_0^\infty e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt
\end{aligned}$$

Intertemporal utility maximization

- Formal mathematical maximization problem:

  $$\max_{c(t)}\left\{B\int_0^\infty e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt\right\},$$

  subject to $k(0)+\int_0^\infty e^{-R(t)}w(t)e^{(g+n)t}dt = \int_0^\infty e^{-R(t)}c(t)e^{(g+n)t}dt$

  and $\lim_{t\to\infty}\left[e^{-R(t)}e^{(n+g)t}k(t)\right]\geq 0.$
  - The last constraint is the "no Ponzi scheme" constraint that prevents households from driving their wealth infinitely negative as time passes.
  - No one would lend to a household that did this, so they wouldn't be able to do it.
- This is a problem in the calculus of variations. We often call this kind of problem "dynamic control theory" in economics.
- We won't explore the solution method in detail in class. (Romer sketches the solution on page 54.)
- The solution for the optimal consumption path consists of two parts
  - The **consumption Euler equation** $\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho-\theta g}{\theta}$, which describes the growth rate of *c* at each point along the consumption path.
    - The Euler equation tells the "slope" (in growth rate terms) of the path at each point, but doesn't determine the level.
  - The **budget constraint** $k(0)+\int_0^\infty e^{-R(t)}w(t)e^{(g+n)t}dt = \int_0^\infty e^{-R(t)}c(t)e^{(g+n)t}dt$ that determines which of the infinite number of parallel consumption paths the household can afford. (It chooses the highest one it can afford.)
  - Romer shows in footnote 9 on page 57 that the initial (time 0) value on the consumption path is

$$c(0)=\frac{k(0)+\int_0^\infty e^{-R(t)}w(t)e^{(g+n)t}\,dt}{\int_0^\infty e^{\frac{(1-\theta)R(t)+(\theta n-\rho)t}{\theta}}\,dt}$$
$$=\frac{k(0)+\int_0^\infty e^{-R(t)}\left[f(k(t))-k(t)f'(k(t))\right]e^{(g+n)t}\,dt}{\int_0^\infty e^{\frac{(1-\theta)R(t)+(\theta n-\rho)t}{\theta}}\,dt}.$$

Note that $R(t)$ depends on $r(\tau)$ at all points in time between 0 and t, and that $r(\tau)=f'(k(\tau))-\delta$ is the net return on capital at time τ.

Thus, $c(0)$ depends only on the future time path of k (and the parameters of the model).

- **Intuition of the Euler equation**
  - Note that c is consumption per effective worker.
    - To get this back to consumption per worker, we use $C = cA$.
    - This means $\frac{\dot{C}(t)}{C(t)}=\frac{\dot{c}(t)}{c(t)}+g=\frac{r(t)-\rho-\theta g}{\theta}+g=\frac{r(t)-\rho}{\theta}$.
  - $\dot{C}>0$ means individuals are choosing a rising consumption path at moment t, so their consumption shortly after t is higher than at t.
  - Correspondingly, $\dot{C}<0$ means that an individual's consumption is lower at a moment after t than at t, and
  - $\dot{C}=0$ means that consumption per person is the same just after t as at t.
  - From the equation:
    - $$r(t)>\rho\Rightarrow\dot{C}>0$$ $$r(t)=\rho\Rightarrow\dot{C}=0$$ $$r(t)<\rho\Rightarrow\dot{C}<0$$
  - These conditions correspond exactly to those of the two-period model:
    - $$r>\rho\Rightarrow C_2>C_1$$ $$r=\rho\Rightarrow C_2=C_1$$ $$r<\rho\Rightarrow C_2<C_1$$
  - The intuition is the same: people will choose a rising, flat, or falling consumption path per person (at moment t) depending on whether the reward to saving (return to capital = interest rate) exceeds, equals, or falls short of their marginal rate of time preference.
  - **How much does a change in r affect the consumption decision?**
    - Change in r is change in slope of budget constraint
    - How far this changes the optimal consumption point depends on amount of curvature in indifference curves
    - Large $\theta\Rightarrow$ lots of curvature; $\theta\to 0$ implies straight line
      - Large θ in denominator means given gap between r and ρ leads to small change in consumption path

        - Large θ means that households do not like to substitute consumption over time: want to stick to smooth consumption path regardless of *r*
        - 1/θ is the elasticity of intertemporal substitution
    - Small θ ⇒ nearly linear indifference curves
        - Small θ in denominator means large change in consumption resulting from given gap between *r* and ρ
        - Small θ means households are OK with substituting over time

Aggregate Dynamics of Ramsey Growth Model

Variables of the model

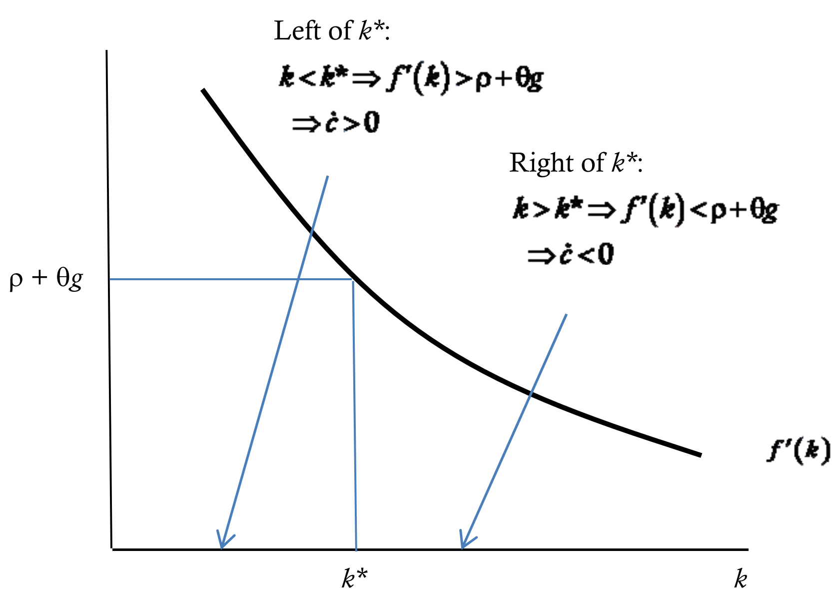
- We analyze the dynamics of the Ramsey model in terms of two variables:
    - Our familiar $k = K/AL$
    - c = Aggregate Consumption $/AL$.
        - Romer uses C for consumption per L, so $c = C/A$.
- These variables differ in their essential nature
    - k is a **state variable** because under the equations of motion of the model it cannot "jump" discontinuously at a moment in time.
        - Note that we could conduct experiments in which k jumped, such as a war that destroys part of the capital stock or a sudden immigration of new workers with no capital. This amounts to momentarily suspending the laws of motion.
    - c is a **control variable** because households could, if they wanted to, change c discretely at a point in time.
        - We will see that households would only do this if they got new information that caused them to revise their lifetime consumption path
        - Along the planned consumption path, they will choose not to have discrete jumps in consumption

Equilibrium rental price of capital

- In equilibrium at every moment, $r(t) = f'(k(t)) - \delta$
    - The real return on loans r must equal the net return on capital $f' - \delta$
    - This links the r in households' budget constraints to conditions in the macroeconomy
    - [Note that this is a growth model and that there are no monetary effects, so the interest rate is the long-run (but not long-term) rate that equilibrates the demand and supply of lending/saving and borrowing/investing independent of any liquidity influences from monetary conditions.]

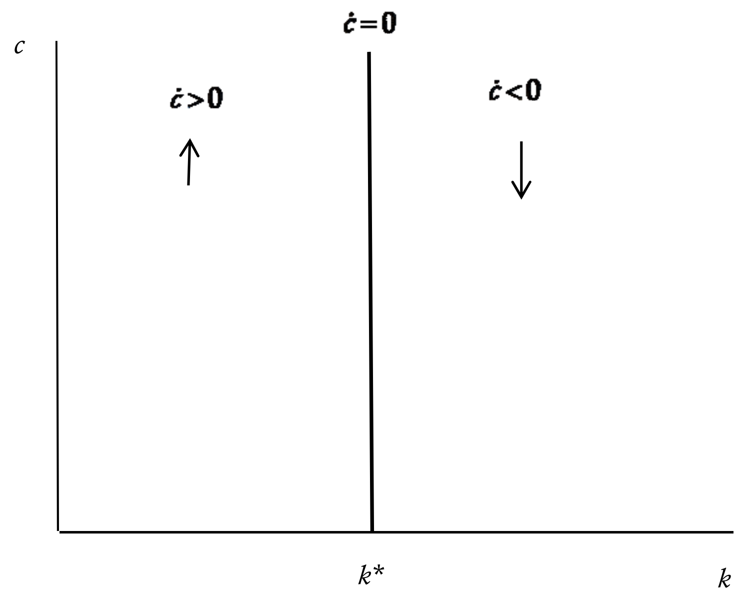
Equations of motion and stationary curves

- We have two equations of motion
  - One from the consumption Euler equation
  - One from the usual capital-stock evolution equation
- Consumption Euler equation: $\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$
  - This equation tells us whether c is rising, falling, or stationary as a function of $k(t)$
  - Note that the level of c does not play a role in determining the sign of $\dot{c}$
  - If $f'(k(t)) > \rho + \theta g$ then $\dot{c} > 0$ because the reward to saving exceeds people's innate preference for current consumption
  - If $f'(k(t)) < \rho + \theta g$ then $\dot{c} < 0$ because the reward to saving falls short of people's innate preference for current consumption
  - If $f'(k(t)) = \rho + \theta g$ then $\dot{c} = 0$ because the reward to saving balances people's innate preference for current consumption
  - Because $f'(k)$ is monotonic, there is a unique value of k at which $f'(k(t)) = \rho + \theta g$:

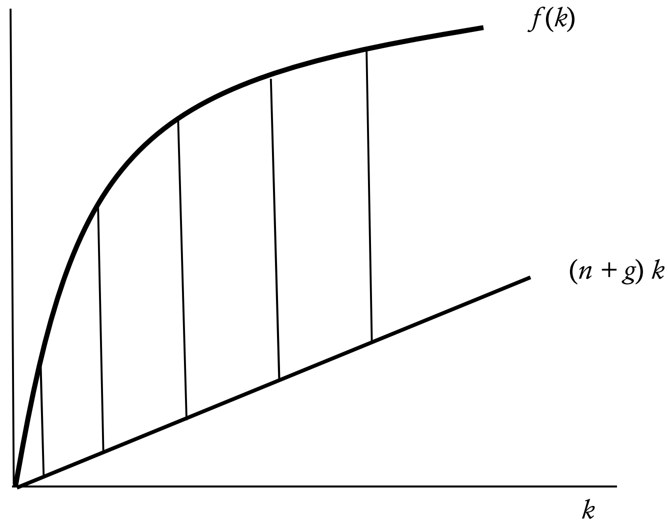


  - At $k = k^*$, $\dot{c} = 0$, although notice that per-capita consumption $C = Ac$ is growing at g
  - We want a "phase diagram" in the dimensions of our two key variables, k and c, so we get the diagram below

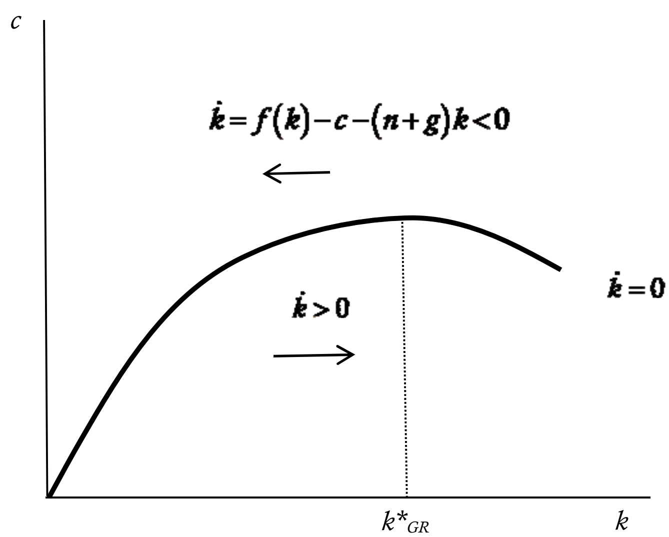
- The arrows in the areas to the left and right of k^* indicate the directions of motion (in the c dimension) if the economy is at a point in those regions.
  - The farther the economy is from the $\dot{c} = 0$ line the more $\dot{c}$ will differ from zero.
  - At points close to k^* c will be nearly stable; at points far from k^* c will be changing rapidly because the interest rate will be far out of line with $\rho + \theta g$.



- Capital accumulation equation: $\dot{k}(t) = f\left[k(t)\right] - c(t) - (n+g)k(t)$
  - First two terms are $y - c$, which is saving per AL, just like $sf(k)$ in Solow model
  - Final term is breakeven investment per AL, just like Solow model with $\delta = 0$
  - What combinations of (k, c) make $\dot{k} = 0$?
    - $\dot{k} = f(k) - c - (n+g)k = 0 \Rightarrow c = f(k) - (n+g)k$
    - The graphical version of this is familiar from the Golden Rule analysis in the Solow model

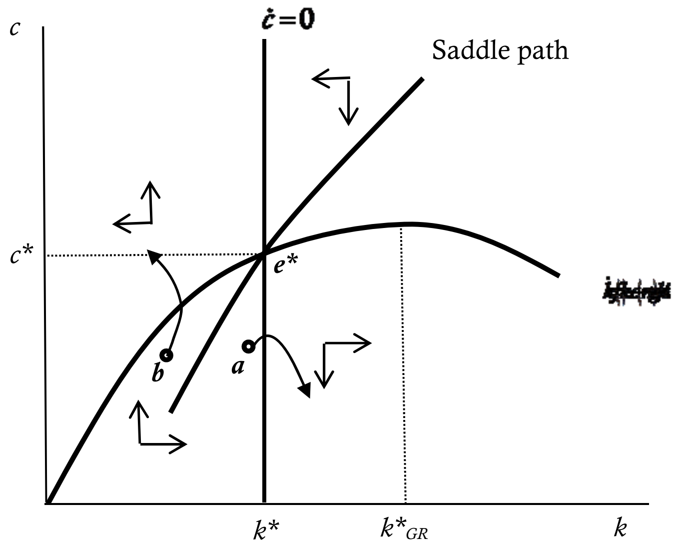


- Gap between curve and line is c that makes $\dot{k} = 0$ at alternative levels of k, which peaks at the Golden Rule value of k^*
- Graphing this in terms of k and c gives



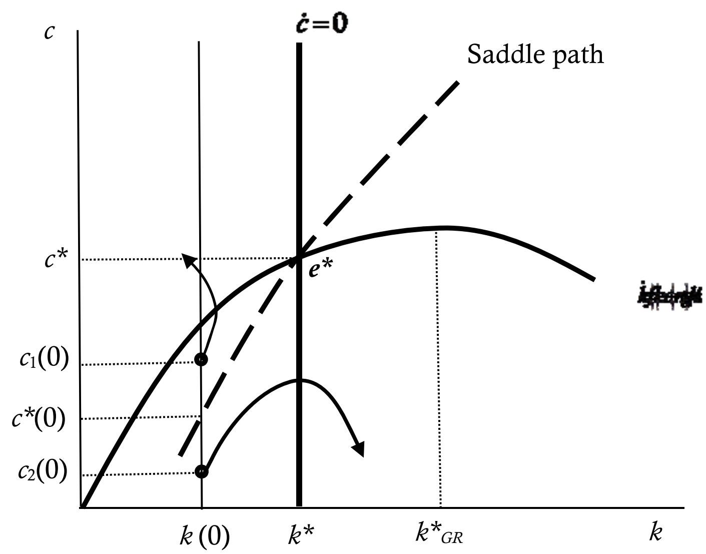
- At combinations of k and c that are above the curve, $\dot{k} < 0$ and at combinations below the curve, $\dot{k} > 0$, so the horizontal arrows shown reflect the movement of k from any point.

* Combining $\dot{c} = 0$ and $\dot{k} = 0$ into a single "phase plane"



- There is a unique equilibrium (k^*, c^*) at which both $\dot{k} = 0$ and $\dot{c} = 0$
- Is it stable?
  - In upper left quadrant, $\dot{k} < 0$ and $\dot{c} > 0$, so economy is diverging away from e^*
  - In lower right quadrant, $\dot{k} > 0$ and $\dot{c} < 0$, so economy is also diverging
  - Lower-left and upper-right seem like they could be stable.
    - Consider point *a*:
      - It is far from the $\dot{k} = 0$ curve so $\dot{k} >> 0$ but close to the $\dot{c} = 0$ line so $\dot{c} \approx 0$.
      - That means that the direction of movement is going to be mostly to the right and only a little bit up.
      - When it hits the line, the direction of motion is directly to the right, so it moves into the region of instability.
    - Similarly point *b*:
      - It is near the $\dot{k} = 0$ curve so $\dot{k} \approx 0$ but far from the $\dot{c} = 0$ line so $\dot{c} >> 0$.
      - This means that economy moves mostly up and only a little to the right.
      - When it hits the line, direction of motion is straight up, moving it into the unstable quadrant and away from e^*
  - There is a single, unique path (the **saddle path**) through e^* along which the economy converges.
    - This is called a "saddle-point" equilibrium because convergence is like rolling a marble down a saddle and getting it to stop at the bottom without rolling off.
- What makes us confident that the economy will find the saddle path and converge?

- At any moment, k is given and households choose c
- The Euler equation reflecting the tradeoff between present and future income is reflected in the behavior of $\dot{c}$ in the phase place.
    - This builds in the slope of the budget constraint through $r = f'(k)$
- However, we have not yet built in the position of the budget constraint through lifetime wealth
    - For any given (current) k, if the consumer sets the consumption path too high, she will exhaust her lifetime income and her path will be unsustainable. This path looks like the one from $c_1(0)$.
    - If she sets the path too low, she will see her capital grow and be inside her budget constraint. This path looks like the one from $c_2(0)$.
    - There is only one path for c (with the shape dictated by the Euler equation) that matches the present value of lifetime consumption with the present value of lifetime resources: it is the one for which the current c lies on the saddle path: $c^*(0)$



    - Thus we are confident that households will choose the consumption level on the saddle path in order to balance their lifetime budget constraints.

Pareto optimality

- All households are identical, so any situation that maximizes the utility of one maximizes all.
- There are no externalities here because one household's consumption decision does not affect anyone else's utility.
- Thus, competitive equilibrium is Pareto optimal.

- Romer shows that a social planner would choose the same consumption path to maximize collective utility as our individual households choose.
- This is application of First Theorem of Welfare Economics: competitive equilibrium (under ideal conditions) leads to Pareto optimal resource allocation

Characteristics of Ramsey Balanced Growth Path

- Once the economy has converged to e^*, how will important variables behave along the steady-state growth path?
  - As in the Solow model, k is constant over time ($\dot{k} = 0$) on steady-state growth path
  - That means $\frac{K}{AL}$ is constant, so K must grow at $n + g$. $y = f(k)$, so k constant means y constant and Y grows at $n + g$.
  - Y/L grows at g.
  - These are exactly the same as the Solow model.
  - Note that $k^* < k_{GR}$.
    - This happens because $\rho > 0$, so households that were in a Golden Rule situation (with steady-state c maximized) would always rather take a little more c now in exchange for a little less in the steady state.
    - They will always choose levels of k^*, y^*, and c^* below those of the Golden Rule because of their impatience.
- How does ρ affect the steady-state equilibrium?
  - If ρ increases, that means that households become less patient
    - We would expect lower saving and a smaller k^*
  - Using the MPK graph, $\rho + \theta g$ moves up, meaning that the level of k at which $f'(k) = \rho + \theta g$ decreases.
  - This shifts the $\dot{c} = 0$ line to the left and a lower k^*
  - The new saddle path will cut the existing level of k above e_0, so consumption jumps instantaneously to the saddle path, then falls along the saddle path to the new equilibrium as their reduced saving erodes k.