

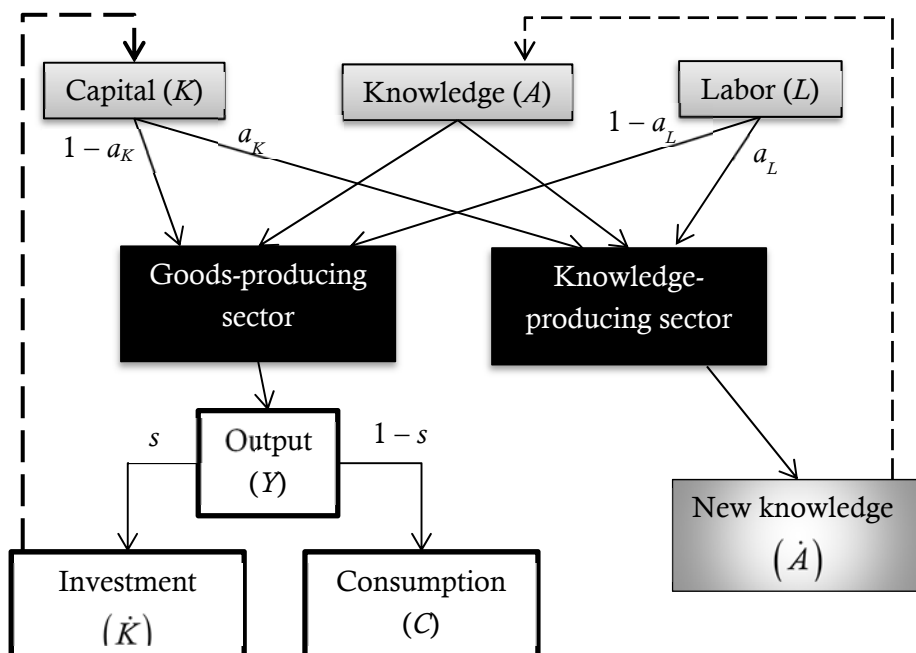
Introduction to Endogenous Growth Models

- Paul Romer's 1986 model and Robert Lucas's (1988) human capital model.
- These models get around the diminishing marginal returns to "capital" assumption by broadening the definition of capital to include knowledge or human capital, both of which may have positive externalities.
- We need some kind of external effects in order to have a model in which
 - individual firms do not have increasing overall returns to scale, so they do not expand infinitely and become economy-wide monopolies
 - the economy as a whole has increasing returns to scale, so that returns to "capital" or "produced inputs" can be constant
- Endogenous growth models have several features that economists have found attractive
 - They endogenize key parameters of the model such as g
 - They can explain lack of convergence
 - They allow s and related policy variables to affect the growth rate of GDP, not just the level of the growth path
- Text book begins with a simplified model of knowledge production via research and development in Chapter 3.
 - Uses constant saving assumption as in Solow model
 - Incorporating Ramsey saving model does not change basic dynamics
- Key characteristic leading to endogenous growth: constant returns to scale in produced inputs.
 - In Solow and Ramsey models, capital was only produced input and had diminishing returns

David Romer's R&D model

Dynamics and behavioral assumptions

- Economy has two sectors: goods-producing sector and R&D (knowledge-producing) sector
- Each sector uses labor and capital
 - a_L and a_K are the shares of labor and capital allocated to the knowledge sector
 - These should be determined by choices of owners of labor and capital allocating them to their highest return
 - Romer simplifies the model by taking these to be exogenous
 - Econ 454 studies models in which the rewards to capital and labor in the two sectors are explicitly modeled and these decisions are allowed to be endogenous



- We assume a Cobb-Douglas CRTS production function for goods:

$$Y(t) = [(1 - a_K)K(t)]^\alpha [A(t)(1 - a_L)L(t)]^{1-\alpha}$$
- Knowledge is produced according to a Cobb-Douglas that may or may not have CRTS:

$$\dot{A}(t) = B[a_K K(t)]^\beta [a_L L(t)]^\gamma A(t)^\theta$$
 - Note that $\theta = 1 \Rightarrow \frac{\dot{A}}{A} = B[a_K K]^\beta [a_L L]^\gamma$ which means growth rate of A (our old g) depends on the amounts of K and L devoted to research and is constant if those amounts are constant.
 - Replication argument cannot be used to justify CRTS here
 - Same knowledge produced by two people is not twice as valuable
 - Positive spillovers could yield increasing returns to scale
 - Are other discoveries substitutes or complements for the next discovery?
- No depreciation and constant saving rate mean $\dot{K}(t) = sY(t)$
- Exogenous growth of labor force: $\dot{L}(t) = nL(t)$

Analysis of R&D Model

- Romer begins with a model in which there is no physical capital ($\alpha = \beta = 0$)
 - We won't analyze this model in detail, but note the equations of the model if $\theta = 1$ and $n = 0$ (so L is constant)

$$Y(t) = A(t)(1 - a_L)L$$

- $\frac{\dot{A}(t)}{A(t)} = B[a_L L]^\gamma$
- Output is proportional to A and the growth rate of A is a constant, so this model has a constant growth rate of output that is determined by B , a_L , and L (and γ).
- Higher R&D productivity, more labor being used in the labs, and a bigger population all lead to a higher growth rate (not just to a higher, parallel growth path)

- For the full model (with K), we have two state variables, A and K
 - We denote the growth rates of A and K by g_A and g_K

- **Dynamics of K**

$$\dot{K}(t) = sY(t) = [s(1 - a_k)^\alpha (1 - a_L)^{1-\alpha}] K(t)^\alpha A(t)^{1-\alpha} L(t)^{1-\alpha}$$

- The term in brackets is a constant (over time) that we shall call c_K
- The growth rate of K at every moment t is

$$g_K(t) \equiv \frac{\dot{K}(t)}{K(t)} = c_K \left[\frac{A(t)L(t)}{K(t)} \right]^{1-\alpha}$$

- We seek a steady state in which K grows at a constant rate g_K^* , so we want to analyze the change in or growth rate of the growth rate
- $\frac{\dot{g}_K(t)}{g_K(t)} = (1 - \alpha)(g_A + n - g_K)$ using our rules for growth rates

$$\left. \begin{array}{l} \dot{g}_K(t) = 0 \text{ if } g_K(t) = g_A(t) + n \\ \dot{g}_K(t) > 0 \text{ if } g_K(t) < g_A(t) + n \\ \dot{g}_K(t) < 0 \text{ if } g_K(t) > g_A(t) + n \end{array} \right\} \Rightarrow \dot{g}_K = 0: g_K^* = g_A^* + n$$

- The $\dot{g}_K = 0$ curve is a line with slope of one and intercept on the g_K axis at $n \geq 0$.

- Below the line, $\dot{g}_K > 0$ and above the line $\dot{g}_K < 0$, so the arrows point vertically toward the line

- **Dynamics of A**

$$g_A(t) = \frac{\dot{A}(t)}{A(t)} = [Ba_K^\beta a_L^\gamma] K(t)^\beta L(t)^\gamma A(t)^{\beta-\gamma}$$

- The term in brackets is constant over time and called c_A

- As above, the growth rate of the growth rate at every moment t is

$$\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1) g_A(t)$$

$$\left. \begin{array}{l} \dot{g}_A(t) = 0 \text{ if } g_K(t) = -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \\ \dot{g}_A(t) > 0 \text{ if } g_K(t) > -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \\ \dot{g}_A(t) < 0 \text{ if } g_K(t) < -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \end{array} \right\} \Rightarrow \dot{g}_A = 0 : g_K^* = \frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A^*(t)$$

- The $\dot{g}_A = 0$ curve is a line with slope $\frac{1-\theta}{\beta}$ and intercept on the vertical (g_K) axis at $-\gamma n/\beta \leq 0$.
- To the left of the line $\dot{g}_A > 0$ and to the right of the line $\dot{g}_A < 0$, so the arrows point horizontally toward the line

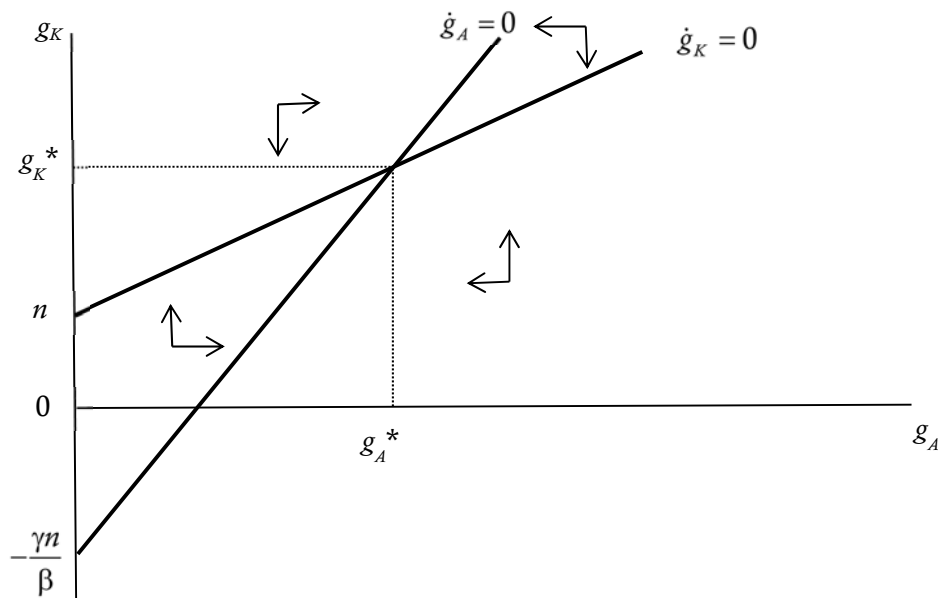
- **Equilibrium dynamics**

- The nature of the equilibrium depends crucially on two properties of the parameters:
 - $n > 0$ vs. $n = 0$
 - This determines whether there is any exogenous source of growth in the model
 - If $n > 0$ as in the Solow and Ramsey models, sustained growth in total GDP is possible through exogenous growth in L
 - $\theta + \beta = 1$ vs. $\theta + \beta < 1$ (or $\theta + \beta > 1$)
 - This determines “returns to scale in produced inputs”
 - Note that K and A are “produced” in the model
 - The production function for goods always has constant returns in produced inputs because K has exponent α and A has exponent $1 - \alpha$
 - The production function for knowledge has returns to scale in the two produced inputs equal to the sum of their exponents: $\theta + \beta$
 - If $\theta + \beta = 1$, then the model can sustain ongoing “endogenous” growth even if $n = 0$ because increases in both K and A together are not subject to diminishing returns

- **Dynamics with $n > 0$ and $n = 1$**

- With $n > 0$, the $\dot{g}_K = 0$ line intercept is positive and the $\dot{g}_A = 0$ line intercept is negative
- If $n = 0$, both lines pass through the origin
- **Case I: $\beta + \theta < 1$** (diminishing returns in produced inputs)

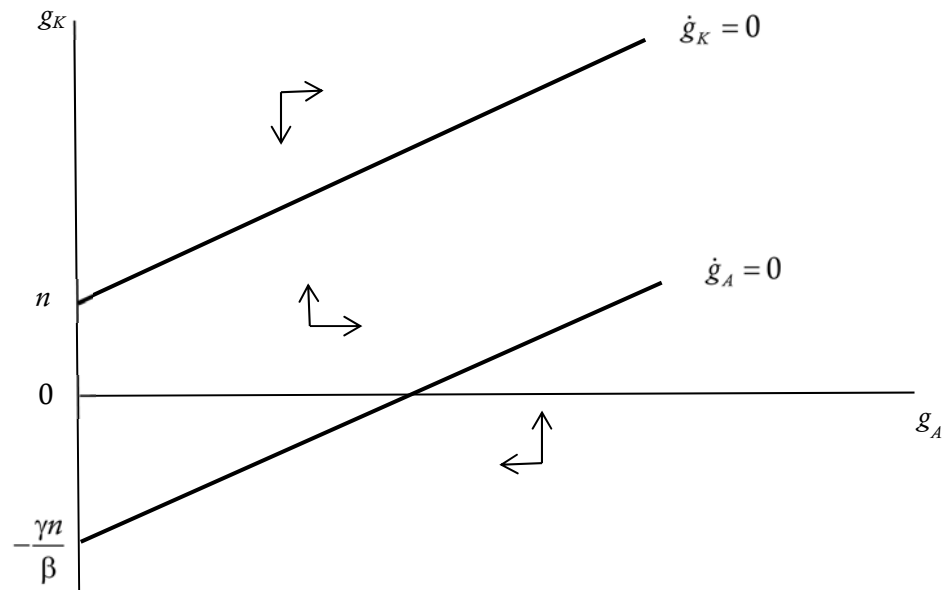
- Slope of $\dot{g}_A = 0$ line is $\frac{1-\theta}{\beta} > 1$, so it is steeper than the $\dot{g}_K = 0$ line



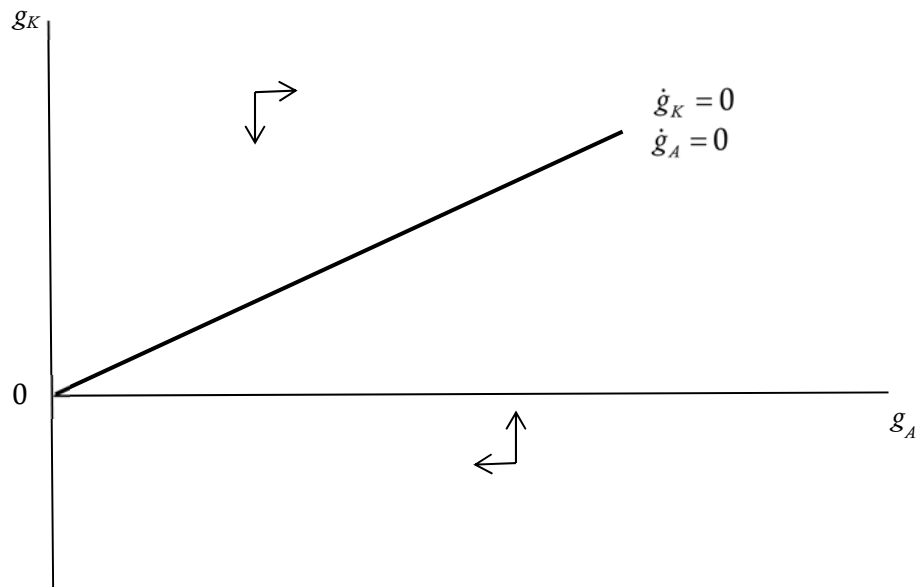
- Economy converges to unique equilibrium from all points in space
- Solving algebraically, we can show

$$g_A^* = \frac{\beta + \gamma}{1 - (\beta + \theta)} n$$

$$g_K^* = \frac{1 - \theta + \gamma}{1 - (\beta + \theta)} n$$
- Growth here is exogenous in the sense that if $n = 0$, both K and A stop growing. (Note that both lines intercept at the origin if $n = 0$.)
- This case replicates the dynamics of the Solow model with g determined endogenously as a function of n
- **Case II: $\beta + \theta = 1$**
 - In this case, the slope of the $\dot{g}_A = 0$ line $\frac{1-\theta}{\beta} = 1$ and the two lines are parallel (or coincident)
 - If $n > 0$, then they are parallel



- The economy will move into the channel between the lines and then growth in both K and A will accelerate forever.
 - Intuitively, a bigger economy means more scientists means more discoveries means faster growth. As long as $n > 0$, the exogenous growth in the labor force leads to accelerating growth.
- If $n = 0$, then the two lines coincide



- Economy converged to the line and on the line, both growth rates are constant and equal (because the line has slope of one)

- Because $g_K^* = g_A^*$, K/A is constant in the steady state
 - There is a unique K/A^* that will sustain equal growth in K and A and a unique common growth rate g^* that is consistent with that K/A^*
 - You will work out the algebra in Problem 3.5.
- Example of this case: Suppose that $B \uparrow$ so that c_A increases.
 - This raises g_A and moves the economy to a point to the right of the original equilibrium.
 - Economy converges back up and to the left to a new equilibrium that is higher than original.
- Endogenous growth occurs in this case: economy sustains positive growth even when there is no exogenous source ($n = 0$)
- Growth rate depends (positively) on s , B , a_K , a_L , and L
- **Case III: $\beta + \theta > 1$**
 - In this case, the $\dot{g}_A = 0$ line is flatter than the $\dot{g}_K = 0$ because $\frac{1-\theta}{\beta} < 1$
 - This case looks like Case II, but the lines are not parallel.
 - In this case, we get explosive growth even when $n = 0$.

Microeconomics of R&D

- The key question that we have dodged in Romer's R&D model: What determines a_K ?
 - Capital owners must decide whether to build factories or labs
 - Economists would assume that they choose the use of their capital that provides the higher rate of return
 - So in equilibrium the amount of capital in the two sectors would have to balance the marginal rates of return
 - Rate of return on factories is straightforward: They produce output that is sold to earn revenue
- How do labs earn money?
 - In the real world, there are lots of funding sources for R&D
 - Corporate funding
 - Government grants
 - Tuition from university students
 - Since we don't model government or university research, we are interested mostly in corporate-funded research and development
 - In our model, knowledge is purely non-rival and non-excludable
 - Any discovery is immediately useful to all producers

- There is no “appropriability” of knowledge for private benefit
 - New knowledge cannot be sold or used profitably
 - Why would capital owners put money into labs that earn nothing?
 - They wouldn’t, so we would need to build a model of how lab owners can earn money from R&D in order to pay for the capital and labor that is used.
- Models of a_K
 - Corporate R&D is profitable if there is an effective way for the company to appropriate the knowledge
 - This usually occurs by preventing other firms from using the knowledge created through some kind of “appropriability mechanism”
 - May also involve licensing
 - Note that either is inefficient, because once created the knowledge is nonrival and “should” be universally used for free
 - Two common appropriability mechanisms are intellectual property rights (patents) and secrecy
 - Both are flawed
 - Some kinds of intellectual property are better protected by patents, some by secrecy, and others are virtually unprotectable
 - Effective patent protection or secrecy gives an effective (but usually temporary) monopoly on the use of the knowledge to the firm doing the R&D
 - Two common models for a_K are based on this:
 - A model of product innovation in which R&D can produce new varieties of (intermediate) goods on which the innovating firm holds a monopoly
 - A model of process innovation in which R&D can advance productive efficiency of one (intermediate) good (of many) and have a cost advantage in production until another firm leap-frogs it
 - Both models add complexity to Romer’s R&D model because both require multiple goods in order to have more than one firm
 - We study both models in Econ 454

Model of Learning by Doing

- Romer’s short section on learning by doing develops the essence of Paul Romer’s first (1986) endogenous growth model.
 - Kenneth Arrow developed a model in the 1960s based on the idea that a firm’s A would be increased as it produced output, so $\dot{A} \sim Y$
 - Paul Romer’s version of this was slightly different
 - Firms’ learning is related to capital accumulation rather than output
 - Knowledge is non-appropriable

- New knowledge occurs as a by-product of capital investment
 - Firms have (some) incentive to invest, so knowledge creation happens despite pure nonrivalry
- Learning by doing with a **constant saving rate**
 - $Y(t) = K(t)^\alpha [A(t)L(t)]^{1-\alpha}$
 - $A(t) = BK(t)^\phi$
 - Solving out A yields $Y(t) = K(t)^\alpha B^{1-\alpha} K(t)^{\phi(1-\alpha)} L(t)^{1-\alpha}$
 $\dot{K}(t) = sY(t) = sB^{1-\alpha} L(t)^{1-\alpha} K(t)^{\alpha+\phi(1-\alpha)}$
 - This model converges, has endogenous growth, or explodes as $\phi < 1$, $\phi = 1$, $\phi > 1$
 - Case of $\phi = 1$ is the endogenous-growth case
 - Let $n = 0$ so there is no exogenous growth
 - $Y(t) = (BL)^{1-\alpha} K(t) \equiv bK(t)$
 $\dot{K}(t) = sY(t) = sbK(t)$
 - $\frac{\dot{K}(t)}{K(t)} = sb$
 - Thus, growth in the capital stock and output is constant at rate sb
 - Any increase in saving, in the productivity of learning, or in the labor force would increase growth
- **Ramsey consumers** in the learning-by-doing model (not done this way in 4th edition)
 - Assume $\phi = 1$ and $n = 0$, so we have the endogenous-growth case
 - Aggregate knowledge is proportional to aggregate capital stock (but this is not the case at the firm level)
 - Firms take aggregate knowledge as given and do not consider how their own investment will add to it because they are small
 - $Y_i(t) = K_i(t)^\alpha [A(t)L_i(t)]^{1-\alpha}$
 - $A(t) = BK(t)$
 $Y_i(t) = B^{1-\alpha} K(t)^{1-\alpha} K_i(t)^\alpha L_i(t)^{1-\alpha}$
 - The private marginal product of capital is

$$\frac{\partial Y_i(t)}{\partial K_i(t)} = \alpha B^{1-\alpha} K(t)^{1-\alpha} \left[\frac{K_i(t)}{L_i(t)} \right]^{-(1-\alpha)} = r(t) \text{ (there is no depreciation)}$$
 - Each firm sets its $\frac{K_i}{L_i}$ so that the private marginal product equals the economy-wide interest rate r

- This means that all firms have the same $\frac{K_i}{L_i}$ that is equal to the aggregate K/L
- Setting $K_i/L_i = K/L$,

$$\begin{aligned}
 r(t) &= \alpha B^{1-\alpha} K(t)^{1-\alpha} \left[\frac{K(t)}{L(t)} \right]^{-(1-\alpha)} \\
 &= \alpha B^{1-\alpha} K(t)^{1-\alpha} K(t)^{-(1-\alpha)} L(t)^{1-\alpha} \\
 &= \alpha [BL(t)]^{1-\alpha} = \alpha [BL]^{1-\alpha} = \bar{r}
 \end{aligned}$$
- This rate of return \bar{r} is constant over time (with $n = 0$) and depends on the rate of knowledge accumulation through investment B , α , and the size of the labor force
- Note that the marginal *social* product of capital (varying K as well as K_i) is larger than the marginal *private* product
 - $MSP_K = \frac{\partial Y_i}{\partial K_i} \Big|_{K_i=K} = (BL)^{1-\alpha} > MPP_K$
 - This means that individual firms will underinvest in capital
 - They do not take into account the positive social externality that their investment conveys on all firms through increased knowledge
 - This means that the privately generated growth rate will be lower than the socially optimal growth rate
- Ramsey consumers, as usual, choose a consumption path that satisfies the Euler equation $\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}$.
 - In this case, $r(t) = \bar{r}$ and $\frac{\dot{C}(t)}{C(t)} = \frac{\bar{r} - \rho}{\theta} = \frac{\alpha(BL)^{1-\alpha} - \rho}{\theta} = \bar{g}$
- To satisfy the economy's budget constraint, Y must grow at the same rate as C , so the economy grows at \bar{g} at every instant.
- There are no convergence dynamics: wherever an economy is, it just grows at \bar{g} from there. (Poor countries with same parameters do not catch up.)
- Growth rate \bar{g} depends on parameters of economy: $\alpha, B, L, \rho, \theta$.
- Once again, we have “scale effects” because a larger L means a faster growth rate.
 - If we allow $n > 0$, then we have both endogenous and exogenous growth and the growth rate accelerates over time.

- Are scale effects realistic? Some argue no, but Kremer's argument for Eurasia, Australia, and Tasmania seems to provide some support.
- In addition, there is much evidence that growth has accelerated over the centuries (as population has grown).
- Non-optimality: social planner would internalize the knowledge externality and use $r^* = (BL)^{1-\alpha} > \bar{r} = \alpha(BL)^{1-\alpha}$ leading to faster growth at $g^* = \frac{(BL)^{1-\alpha} - \rho}{\theta} > \bar{g}$

(Paul) Romer model (not worth doing the details)

- What's different about this model?
 - We model the incentives for production of knowledge explicitly
 - We introduce the "Ethier production function" and the now-ubiquitous model of a continuum of "intermediate goods"

Human Capital in the Solow Model

- Distinction between knowledge capital and human capital
 - Latter is rival and embodied in worker
 - Former relates to nonrival ideas that all share (costlessly)
- Model is motivated by the dominant question: "Why are some countries richer than others?"
 - Solow model says differences in k
 - Not plausible (as Romer shows late in Ch 1)
 - Mankiw, Romer, & Weil: differences in physical and human capital
 - They argue this is plausible; others disagree
 - Differences in A
 - Why would technology be different across countries?
 - Barriers (legal and otherwise) to adoption
 - Non-applicability of advanced technologies in poor countries (climate, unreliable physical infrastructure, etc.)
 - Differences in "social infrastructure"
 - We'll have more to say about this soon
- How to incorporate human capital into model?
 - Many alternative ways; Romer does one (and others in problems 4.8 and 4.9)
 - How does economy "produce" human capital?
 - Process of education or training has two major costs: teachers' time (for which they are paid) and students' time (for which they are not paid)

- Can use a two-sector model with a production function for education using labor (teachers) and capital (schools) like the one for knowledge in the R&D model
- Can just deduct some amount of a conglomerate “output” as being education in a one-sector model (like some output is physical capital rather than consumption). This is Romer’s 4.8.
- Can model the process as holding people out of the labor force during an education period. This is Romer’s Section 4.1.
 - This doesn’t model the cost of teachers and schools.
 - Note that forgone earnings may be higher than teacher/school costs at most schools (if maybe not at Reed)

Simple human-capital model setup

- Let $H(t) \equiv L(t)G(E)$ be the amount of human capital, which is the number of workers $L(t)$ times the amount of human capital per worker $G(E)$, where E is the average education level of current workers.
 - $G'(E) > 0$
 - $G(E) = e^{\phi E}$ is a commonly used functional form
 - We assume that in a steady state with education level E , people live T years, going to school for E years and working for $T - E$ years.
 - In general (but not in this model), human capital includes not just education but training, health and other “acquired” characteristics that affect labor productivity.
- $Y(t) = K(t)^\alpha [A(t)H(t)]^{1-\alpha}$
- $\dot{K}(t) = sY(t) - \delta K(t)$
- $\dot{A}(t) = gA(t)$
- $\dot{L}(t) = nL(t)$

Solving the model

- This model looks (and behaves) similarly to Solow model
- Define $k \equiv \frac{K}{AH} = \frac{K}{ALG(E)}$
- $\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t)$
- $\dot{k} = 0 \Rightarrow k = k^* = \left(\frac{s}{n + g + \delta} \right)^{\frac{1}{1-\alpha}}$

- How will a **change in E** affect the steady-state growth path?
 - Effects of $E \uparrow$ (or $G \uparrow$) on K and Y are equivalent to increase in L
 - Economy moves to higher, parallel steady-state path
 - Level effect, but no growth effect
 - Y and Y/L are higher in steady-state
- But the important variable (living standards) here is Y/N , where N is total population
 - $\left(\frac{Y(t)}{N(t)}\right)^* = y^* A(t) G(E) \left(\frac{L(t)}{N(t)}\right)^*$ on the steady-state path
 - Increase in E does not affect y^* or $A(t)$
 - Increase in E raises $G(E)$
 - Increase in E lowers L/N because more people are in school and fewer in the labor force
 - What will be the net effect?
 - What is L/N ?
 - It seems like it should be $(T-E)/T$ since that is the ratio of working years to total life years for each individual
 - That is correct if $n = 0$
 - If the population is growing, then the cohort in education is larger than the cohort that is working.
 - Romer (and Coursebook) shows that in steady state

$$\frac{L(t)}{N(t)} = \frac{e^{-nE} - e^{-nT}}{1 - e^{-nT}}$$
 - It is intuitively clear (and mathematically easy) that $\frac{\partial(L/N)}{\partial E} < 0$

Dynamics of increase in E

- Initial effect lowers Y because fewer people in labor force but no immediate increase in the education of those who are working
- In steady state, the two effects noted above are in conflict and we don't know which will dominate
 - $\frac{\partial(Y/N)}{\partial E} = \frac{\partial(Y/N)}{\partial E} + \frac{\partial(Y/N)}{\partial(L/N)} \frac{\partial(L/N)}{\partial E}$
 - The first term depends mostly on $G'(E)$ and the second is negative.
 - If $G'(E)$ is large, then Y/N is likely to rise with an increase in E

- This makes intuitive sense: if education is highly productive it will raise per-capita income; if it is not, then it drains people who could be working into useless education.