Introduction to Endogenous Growth Models

- These models get around the diminishing marginal returns to “capital” assumption by broadening the definition of capital to include knowledge or human capital, both of which may have positive externalities.
- We need some kind of external effects in order to have a model in which
  - individual firms do not have increasing overall returns to scale, so they do not expand infinitely and become economy-wide monopolies
  - the economy as a whole has increasing returns to scale, so that returns to “capital” or “produced inputs” can be constant
- Endogenous growth model have several features that economists have found attractive
  - They endogenize key parameters of the model such as $g$
  - They can explain lack of convergence
  - They allow $s$ and related policy variables to affect the growth rate of GDP, not just the level of the growth path
- Text book begins with a simplified model of knowledge production via research and development in Chapter 3.
  - Uses constant saving assumption as in Solow model
  - Incorporating Ramsey saving model does not change basic dynamics
- Key characteristic leading to endogenous growth: constant returns to scale in produced inputs.
  - In Solow and Ramsey models, capital was only produced input and had diminishing returns

David Romer’s R&D model

Dynamics and behavioral assumptions

- Economy has two sectors: goods-producing sector and R&D (knowledge-producing) sector
- Each sector uses labor and capital
  - $a_L$ and $a_K$ are the shares of labor and capital allocated to the knowledge sector
  - These should be determined by choices of owners of labor and capital allocating them to their highest return
  - Romer simplifies the model by taking these to be exogenous
  - Econ 454 studies models in which the rewards to capital and labor in the two sectors are explicitly modeled and these decisions are allowed to be endogenous
We assume a Cobb-Douglas CRTS production function for goods:
\[ Y(t) = [(1-a_K)K(t)]^\alpha [A(t)(1-a_L)L(t)]^{1-\alpha} \]
Knowledge is produced according to a Cobb-Douglas that may or may not have CTRS:
\[ \dot{A}(t) = B[a_K K(t)]^\beta [a_L L(t)]^\gamma A(t)^\theta \]
- Note that \( \theta = 1 \Rightarrow \frac{\dot{A}}{A} = B[a_K K(t)]^\beta [a_L L(t)]^\gamma \) which means growth rate of \( A \) (our old \( g \)) depends on the amounts of \( K \) and \( L \) devoted to research and is constant if those amounts are constant.
- Replication argument cannot be used to justify CRTS here
  - Same knowledge produced by two people is not twice as valuable
  - Positive spillovers could yield increasing returns to scale
  - Are other discoveries substitutes or complements for the next discovery?

No depreciation and constant saving rate mean \( \dot{K}(t) = sY(t) \)
Exogenous growth of labor force: \( \dot{L}(t) = nL(t) \)

**Analysis of R&D Model**
- Romer begins with a model in which there is no physical capital (\( \alpha = \beta = 0 \))
  - We won’t analyze this model in detail, but note the equations of the model if \( \theta = 1 \) and \( n = 0 \) (so \( L \) is constant)
David Romer’s R&D model

\[ Y(t) = A(t)(1 - a_e)L \]

- \[ \frac{\dot{A}(t)}{A(t)} = B[a_L L] \]

- Output is proportional to \( A \) and the growth rate of \( A \) is a constant, so this model has a constant growth rate of output that is determined by \( B, a_L, \) and \( L \) (and \( \gamma \)).

- Higher R&D productivity, more labor being used in the labs, and a bigger population all lead to a higher growth rate (not just to a higher, parallel growth path)

- For the full model (with \( K \)), we have two state variables, \( A \) and \( K \)
  - We denote the growth rates of \( A \) and \( K \) by \( g_A \) and \( g_K \)

**Dynamics of \( K \)**

\[ \dot{K}(t) = sY(t) = \left[ s(1 - a_e)(1 - a_e)^{1-a_L} \right] K(t)^{1-a} A(t)^{1-a_L} L(t)^{1-a_K} \]

- The term in brackets is a constant (over time) that we shall call \( c_K \)
- The growth rate of \( K \) at every moment \( t \) is
  \[ g_K(t) = \frac{\dot{K}(t)}{K(t)} = c_K \left[ \frac{A(t)L(t)}{K(t)} \right]^{1-a} \]

- We seek a steady state in which \( K \) grows at a constant rate \( * \)
- The term \( \frac{\dot{g}_K(t)}{g_K(t)} = (1 - \alpha)(g_A + n - g_K) \) using our rules for growth rates
  - \( \dot{g}_K(t) = 0 \) if \( g_K(t) = g_A(t) + n \)
  - \( \dot{g}_K(t) > 0 \) if \( g_K(t) < g_A(t) + n \) \( \Rightarrow \dot{g}_K = 0 \) \( \Rightarrow g_K = g_A + n \)
  - \( \dot{g}_K(t) < 0 \) if \( g_K(t) > g_A(t) + n \)
- The \( \dot{g}_K = 0 \) curve is a line with slope of one and intercept on the \( g_K \) axis at \( n \geq 0 \).
  - Below the line, \( \dot{g}_K > 0 \) and above the line \( \dot{g}_K < 0 \), so the arrows point vertically toward the line

**Dynamics of \( A \)**

\[ g_A(t) = \frac{\dot{A}(t)}{A(t)} = [Bd_L^0 a_L^0] K(t)^0 L(t)^0 A(t)^{0-1} \]

- The term in brackets is constant over time and called \( c_A \)
As above, the growth rate of the growth rate at every moment $t$ is

\[
\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_k(t) + \gamma n + (\theta - 1) g_A(t)
\]

- \( \dot{g}_A(t) = 0 \) if \( g_k(t) = -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \)
- \( \dot{g}_A(t) > 0 \) if \( g_k(t) > -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \) \( \Rightarrow \dot{g}_A = 0 : g_k^* = -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \)
- \( \dot{g}_A(t) < 0 \) if \( g_k(t) < -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t) \)

- The \( \dot{g}_A = 0 \) curve is a line with slope \( \frac{1-\theta}{\beta} \) and intercept on the vertical \( (g_k) \) axis at \( -\gamma n/\beta \leq 0 \).

- To the left of the line \( \dot{g}_A > 0 \) and to the right of the line \( \dot{g}_A < 0 \), so the arrows point horizontally toward the line.

- **Equilibrium dynamics**
  - The nature of the equilibrium depends crucially on two properties of the parameters:
    - \( n > 0 \) vs. \( n = 0 \)
      - This determines whether there is any exogenous source of growth in the model
      - If \( n > 0 \) as in the Solow and Ramsey models, sustained growth in total GDP is possible through exogenous growth in \( L \)
    - \( \theta + \beta = 1 \) vs. \( \theta + \beta < 1 \) (or \( \theta + \beta > 1 \))
      - This determines “returns to scale in produced inputs”
      - Note that \( K \) and \( A \) are “produced” in the model
      - The production function for goods always has constant returns in produced inputs because \( K \) has exponent \( \alpha \) and \( A \) has exponent \( 1 - \alpha \)
      - The production function for knowledge has returns to scale in the two produced inputs equal to the sum of their exponents: \( \theta + \beta \)
      - If \( \theta + \beta = 1 \), then the model can sustain ongoing “endogenous” growth even if \( n = 0 \) because increases in both \( K \) and \( A \) together are not subject to diminishing returns

- **Dynamics with \( n > 0 \) and \( n = 0 \)**
  - With \( n > 0 \), the \( \dot{g}_k = 0 \) line intercept is positive and the \( \dot{g}_A = 0 \) line intercept is negative
  - If \( n = 0 \), both lines pass through the origin
  - **Case I:** \( \beta + \theta < 1 \) (diminishing returns in produced inputs)
• Slope of $\dot{g}_A = 0$ line is $\frac{1-0}{\beta} > 1$, so it is steeper than the $\dot{g}_K = 0$ line

- Economy converges to unique equilibrium from all points in space
- Solving algebraically, we can show
  
  \[
  g^*_A = \frac{\beta + \gamma}{1-(\beta + \theta)} n
  \]
  
  \[
  g^*_K = \frac{1-0 + \gamma}{1-(\beta + \theta)} n
  \]

- Growth here is exogenous in the sense that if $n = 0$, both $K$ and $A$ stop growing. (Note that both lines intercept at the origin if $n = 0$.)
- This case replicates the dynamics of the Solow model with $g$ determined endogenously as a function of $n$

  o **Case II: $\beta + \theta = 1$**

  - In this case, the slope of the $\dot{g}_A = 0$ line $\frac{1-0}{\beta} = 1$ and the two lines are parallel (or coincident)
  - If $n > 0$, then they are parallel
- The economy will move into the channel between the lines and then growth in both $K$ and $A$ will accelerate forever.
- Intuitively, a bigger economy means more scientists means more discoveries means faster growth. As long as $n > 0$, the exogenous growth in the labor force leads to accelerating growth.
- If $n = 0$, then the two lines coincide

- Economy converged to the line and on the line, both growth rates are constant and equal (because the line has slope of one)
• Because \( g_A^* = g_K^* \), \( K/A \) is constant in the steady state
  o There is a unique \( K/A^* \) that will sustain equal growth in \( K \) and \( A \) and a unique common growth rate \( g^* \) that is consistent with that \( K/A^* \)
  o You will work out the algebra in Problem 3.5.

• Examples of this case: Suppose that \( B \uparrow \) so that \( c_A \) increases.
  o This raises \( g_A \) and moves the economy to a point to the right of the original equilibrium.
  o Economy converges back up and to the left to a new equilibrium that is higher than original.

• Second example: \( s \downarrow \) so that \( c_K \) increases
  o Raises \( g_K \) and moves upward above original equilibrium
  o Economy converges down to the right to a new high growth rate

• Third example: \( a_K \uparrow \) so that \( c_K \) falls and \( c_A \) increases
  o Economy moves down and to the right
  o Converges back to line, but could be higher or lower growth rate
  o Change in growth rate depends on the productivity of \( A \) vs. \( K \) at the margin.

• Endogenous growth occurs in this case: economy sustains positive growth even when there is no exogenous source \((n = 0)\)

• Growth rate depends (positively) on \( s, B, a_K, a_L \), and \( L \)
  o Case III: \( \beta + \theta > 1 \)
    ▪ In this case, the \( \dot{g}_A = 0 \) line is flatter than the \( \dot{g}_K = 0 \) because \( \frac{1-\theta}{\beta} < 1 \)
    ▪ This case looks like Case II, but the lines are not parallel.
    ▪ In this case, we get explosive growth even when \( n = 0 \).

### Microeconomics of R&D

• The key question that we have dodged in Romer’s R&D model: What determines \( a_K \)?
  o Capital owners must decide whether to build factories or labs
  o Economists would assume that the choose the use of their capital that provides the higher rate of return
    ▪ So in equilibrium the amount of capital in the two sectors would have to balance the marginal rates of return
Rate of return on factories is straightforward: They produce output that is sold to earn revenue.

How do labs earn money?
- In the real world, there are lots of funding sources for R&D
  - Corporate funding
  - Government grants
  - Tuition from university students
  - Since we don’t model government or university research, we are interested mostly in corporate-funded research and development
- In our model, knowledge is purely non-rival and non-excludable
  - Any discovery is immediately useful to all producers
  - There is no “appropriability” of knowledge for private benefit
  - New knowledge cannot be sold or used profitably
- Why would capital owners put money into labs that earn nothing?
  - They wouldn’t, so we would need to build a model of how lab owners can earn money from R&D in order to pay for the capital and labor that is used.

Models of $a_K$
- Corporate R&D is profitable if there is an effective way for the company to appropriate the knowledge
- This usually occurs by preventing other firms from using the knowledge created through some kind of “appropriability mechanism”
  - May also involve licensing
  - Note that either is inefficient, because once created the knowledge is nonrival and “should” be universally used for free
- Two common appropriability mechanisms are intellectual property rights (patents) and secrecy
  - Both are flawed
  - Some kinds of intellectual property are better protected by patents, some by secrecy, and others are virtually unprotectable
- Effective patent protection or secrecy gives an effective (but usually temporary) monopoly on the use of the knowledge to the firm doing the R&D
- Two common models for $a_K$ are based on this:
  - A model of product innovation in which R&D can produce new varieties of (intermediate) goods on which the innovating firm holds a monopoly
  - A model of process innovation in which R&D can advance productive efficiency of one (intermediate) good (of many) and have a cost advantage in production until another firm leap-frogs it
  - Both models add complexity to Romer’s R&D model because both require multiple goods in order to have more than one firm
We study both models in Econ 454

Model of Learning by Doing

- Romer’s short section on learning by doing develops the essence of Paul Romer’s first (1986) endogenous growth model.
  - Kenneth Arrow developed a model in the 1960s based on the idea that a firm’s $A$ would be increased as it produced output, so $\dot{A} \sim Y$
  - Paul Romer’s version of this was slightly different
    - Firms’ learning is related to capital accumulation rather than output
    - Knowledge is non-appropriable
      - New knowledge occurs as a by-product of capital investment
      - Firms have (some) incentive to invest, so knowledge creation happens despite pure nonrivalry
  - Learning by doing with a constant saving rate
    \[
    Y(t) = K(t)^\alpha \left[ A(t)L(t) \right]^{1-\alpha}
    \]
    - $A(t) = BK(t)^{\phi}$
    - Solving out $A$ yields
      \[
      Y(t) = K(t)^\alpha B^{1-\alpha} K(t)^{\phi(1-\alpha)} L(t)^{1-\alpha}
      \]
      - $\dot{K}(t) = sY(t) = sB^{1-\alpha} L(t)^{1-\alpha} K(t)^{\alpha+\phi(1-\alpha)}$
    - This model converges, has endogenous growth, or explodes as $\phi < 1$, $\phi = 1$, $\phi > 1$
    - Case of $\phi = 1$ is the endogenous-growth case
      - Let $n = 0$ so there is no exogenous growth
      - $Y(t) = (BL)^{1-\alpha} K(t) = bK(t)$
      - $\dot{K}(t) = sY(t) = sbK(t)$
      - $\frac{\dot{K}(t)}{K(t)} = sb$
      - Thus, growth in the capital stock and output is constant at rate $sb$
      - Any increase in saving, in the productivity of learning, or in the labor force would increase growth
  - Ramsey consumers in the learning-by-doing model (not done this way in 4th edition)
    - Assume $\phi = 1$ and $n = 0$, so we have the endogenous-growth case
    - Aggregate knowledge is proportional to aggregate capital stock (but this is not the case at the firm level)
      - Firms take aggregate knowledge as given and do not consider how their own investment will add to it because they are small
\( Y_i(t) = K_i(t)^\alpha \left[ A(t)L_i(t) \right]^{1-\alpha} \)

- \( A(t) = BK(t) \)
- \( Y_i(t) = B^{1-\alpha} K(t)^{1-\alpha} K_i(t)^\alpha L_i(t)^{1-\alpha} \)

- The private marginal product of capital is

\[
\frac{\partial Y_i(t)}{\partial K_i(t)} = \alpha B^{1-\alpha} K(t)^{1-\alpha} \left[ \frac{K_i(t)}{L_i(t)} \right]^{\gamma(1-\alpha)} = r(t) \quad (\text{there is no depreciation})
\]

- Each firm sets its \( \frac{K_i}{L_i} \) so that the private marginal product equals the economy-wide interest rate \( r \)
- This means that all firms have the same \( \frac{K_i}{L_i} \) that is equal to the aggregate \( K/L \)
- Setting \( \frac{K_i}{L_i} = K/L \),

\[
r(t) = \alpha B^{1-\alpha} K(t)^{1-\alpha} \left[ \frac{K(t)}{L(t)} \right]^{\gamma(1-\alpha)} = \alpha B^{1-\alpha} K(t)^{1-\alpha} L(t)^{1-\alpha} = \alpha [BL(t)]^{\gamma-\alpha} = \bar{r}
\]

- This rate of return \( \bar{r} \) is constant over time (with \( n = 0 \)) and depends on the rate of knowledge accumulation through investment \( B, \alpha \), and the size of the labor force.
- Note that the marginal social product of capital (varying \( K \) as well as \( K_i \)) is larger than the marginal private product
  - \( MSP_K = \frac{\partial Y_i}{\partial K_i} \bigg|_{K_i=K} = (BL)^{1-\alpha} > MPP_K \)
  - This means that individual firms will underinvest in capital
  - They do not take into account the positive social externality that their investment conveys on all firms through increased knowledge
  - This means that the privately generated growth rate will be lower than the socially optimal growth rate
- Ramsey consumers, as usual, choose a consumption path that satisfies the Euler equation \( \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta} \).
  - In this case, \( r(t) = \bar{r} \) and \( \frac{\dot{C}(t)}{C(t)} = \frac{\bar{r} - \rho}{\theta} = \frac{\alpha (BL)^{1-\alpha} - \rho}{\theta} = \bar{g} \)
To satisfy the economy’s budget constraint, $Y$ must grow at the same rate as $C$, so the economy grows at $\bar{g}$ at every instant.

There are no convergence dynamics: wherever an economy is, it just grows at $\bar{g}$ from there. (Poor countries with same parameters do not catch up.)

Growth rate $\bar{g}$ depends on parameters of economy: $\alpha$, $B$, $L$, $\rho$, $\theta$.

Once again, we have “scale effects” because a larger $L$ means a faster growth rate.

- If we allow $n > 0$, then we have both endogenous and exogenous growth and the growth rate accelerates over time.
- Are scale effects realistic? Some argue no, but Kremer’s argument for Eurasia, Australia, and Tasmania seems to provide some support.
- In addition, there is much evidence that growth has accelerated over the centuries (as population has grown).

Non-optimality: social planner would internalize the knowledge externality and use $r^* = (BL)^{1-a} > \bar{r} = \alpha (BL)^{1-a}$ leading to faster growth at $g^* = \frac{(BL)^{1-a} - \rho}{\theta} > \bar{g}$

(Paul) Romer model (not worth doing the details)

- What’s different about this model?
  - We model the incentives for production of knowledge explicitly
  - We introduce the “Ethier production function” and the now-ubiquitous model of a continuum of “intermediate goods”

Human Capital in the Solow Model

- Distinction between knowledge capital and human capital
  - Latter is rival and embodied in worker
  - Former relates to nonrival ideas that all share (costlessly)
- Model is motivated by the dominant question: “Why are some countries richer than others?”
  - Solow model says differences in $k$
    - Not plausible (as Romer shows late in Ch 1)
  - Mankiw, Romer, & Weil: differences in physical and human capital
    - They argue this is plausible; others disagree
  - Differences in $A$
    - Why would technology be different across countries?
    - Barriers (legal and otherwise) to adoption
• Non-applicability of advanced technologies in poor countries (climate, unreliable physical infrastructure, etc.)
  o Differences in “social infrastructure”
  ▪ We’ll have more to say about this soon

• How to incorporate human capital into model?
  o Many alternative ways; Romer does one (and others in problems 4.8 and 4.9)
  o How does economy “produce” human capital?
    ▪ Process of education or training has two major costs: teachers’ time (for which they are paid) and students’ time (for which they are not paid)
    ▪ Can use a two-sector model with a production function for education using labor (teachers) and capital (schools) like the one for knowledge in the R&D model
    ▪ Can just deduct some amount of a conglomerate “output” as being education in a one-sector model (like some output is physical capital rather than consumption). This is Romer’s 4.8.
    ▪ Can model the process as holding people out of the labor force during an education period. This is Romer’s Section 4.1.
      • This doesn’t model the cost of teachers and schools.
      • Note that forgone earnings may be higher than teacher/school costs at most schools (if maybe not at Reed)

**Simple human-capital model setup**

• Let $H(t) \equiv L(t)G(E)$ be the amount of human capital, which is the number of workers $L(t)$ times the amount of human capital per worker $G(E)$, where $E$ is the average education level of current workers.
  o $G'(E) > 0$
  o $G(E) = e^{kE}$ is a commonly used functional form
  o We assume that in a steady state with education level $E$, people live $T$ years, going to school for $E$ years and working for $T - E$ years.
  o In general (but not in this model), human capital includes not just education but training, health and other “acquired” characteristics that affect labor productivity.

  • $Y(t) = K(t)^{\alpha} [A(t)H(t)]^{1-\alpha}$
  • $\dot{K}(t) = sY(t) - \delta K(t)$
  • $\dot{A}(t) = gA(t)$
  • $\dot{L}(t) = nL(t)$
Solving the model

- This model looks (and behaves) similarly to Solow model
- Define \( k = \frac{K}{AH} = \frac{K}{ALG(E)} \)
- \( \dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t) \)
- \( \dot{k} = 0 \Rightarrow k = k^* = \left( \frac{s}{n + g + \delta} \right)^{\frac{1}{1-a}} \)
- How will a change in \( E \) affect the steady-state growth path?
  - Effects of \( E \uparrow \) (or \( G \uparrow \)) on \( K \) and \( Y \) are equivalent to increase in \( L \)
  - Economy moves to higher, parallel steady-state path
  - Level effect, but no growth effect
  - \( Y \) and \( Y/L \) are higher in steady-state
- But the important variable (living standards) here is \( Y/N \), where \( N \) is total population
  - \( \left( \frac{Y(t)}{N(t)} \right)^* = y^* A(t)G(E) \left( \frac{L(t)}{N(t)} \right)^* \) on the steady-state path
    - Increase in \( E \) does not affect \( y^* \) or \( A(t) \)
    - Increase in \( E \) raises \( G(E) \)
    - Increase in \( E \) lowers \( L/N \) because more people are in school and fewer in the labor force
    - What will be the net effect?
  - What is \( L/N? \)
    - It seems like it should be \( (T - E) / T \) since that is the ratio of working years to total life years for each individual
    - That is correct if \( n = 0 \)
    - If the population is growing, then the cohort in education is larger than the cohort that is working.
    - Romer (and Coursebook) shows that in steady state
      \[ \frac{L(t)}{N(t)} = \frac{e^{-nE} - e^{-nT}}{1 - e^{-nT}} \]
      - It is intuitively clear (and mathematically easy) that \( \frac{\partial(L/N)}{\partial E} < 0 \)

Dynamics of increase in \( E \)

- Initial effect lowers \( Y \) because fewer people in labor force but no immediate increase in the education of those who are working
In steady state, the two effects noted above are in conflict and we don’t know which will dominate:

\[
\frac{\partial (Y/N)}{\partial E} = \frac{\partial (Y/N)}{\partial E} \frac{\partial (Y/N)}{\partial (L/N)} \frac{\partial (L/N)}{\partial E}
\]

- The first term depends mostly on \(G'(E)\) and the second is negative.
- If \(G'(E)\) is large, then \(Y/N\) is likely to rise with an increase in \(E\).
- This makes intuitive sense: if education is highly productive it will raise per-capita income; if it is not, then it drains people who could be working into useless education.