The Diamond Model: Overlapping Generations

Differences from the Ramsey model
- Discrete time
- Finite (two-period) lifetimes
- Distinct phases of life (work and retirement)

Similarities to Ramsey model
- Same basic idea of consumption/saving
- Same production and growth dynamics

Dynamic assumptions
- Lifetime assumptions
  - Individual lives two periods
  - No links to earlier or later generations
  - Works first period and lives off saving (with interest) second period
  - Old gradually sell off capital to the young throughout the old-age period
  - Young save by buying capital from old, then diverting output from consumption to create additional new capital as desired
- Size of population
  - $L_t = \text{number of young in year } t$
    - They are the only workers
  - $L_{t+1} = (1 + n) L_t$
    - Labor force and population grow at annually compounded rate $n$
- Consumption notation
  - $C_{1,t} = \text{consumption per person by young in period } t$
  - $C_{2,t} = \text{consumption per person by old in period } t$

Utility
- Period felicity function is again CRRA
- Consumption choices by person who is young in $t$
  - Consumes $C_{1,t}$ when young and $C_{2,t+1}$ when old
- Utility is $U = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2,t+1}^{1-\theta}}{1-\theta}$, with $\theta > 0$ and $\rho > 0$.
  - Utility is discounted at annually compounded rate $\rho$
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Production and dynamics

- \( Y_t = F( K_t, A_t L_t ) \) with constant returns to scale and usual marginal product conditions
  - \( y_t = f( k_t ) \) as before
  - \( r_t = f'( k_t ) \) (assume no depreciation)
  - \( w_t = f( k_t ) - k_t f'( k_t ) = \) wage of each effective labor unit
    - \( A_t w_t = \) wage per worker
    - \( A_t w_t L_t = \) total wages earned in economy (= income of young)
- All capital is owned by the old at the beginning of each period
- \( K_{t+1} = L_t ( w_t A_t - C_t ) \)
  - Expression in parentheses is saving by each young person, which consists of buying up capital from oldies plus perhaps diverting some new production to further capital.
  - Multiply by number of young people to get total capital put away for next period

Budget constraint and utility maximization

- Budget constraint over lifetime is \( C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = A_t w_t \)
  - Or \( C_{2,t+1} = (1 + r_{t+1}) ( w_t A_t - C_{1,t} ) \)
- Individual maximization problem:
  \[
  \max_{C_{1,t}, C_{2,t+1}} \left[ C_{1,t}^{1-\theta} + \frac{1}{1+\rho} C_{2,t+1}^{1-\theta} \right], \text{ subject to } C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = A_t w_t;
  \]
- Can do this as a Lagrangean, but just as easy to solve and substitute for \( C_{2,t+1} \): 
  \[
  \max_{C_{1,t}} \left[ C_{1,t}^{1-\theta} + \frac{1}{1+\rho} \left( \frac{(1 + r_{t+1}) ( w_t A_t - C_{1,t} )}{1 - \theta} \right)^{1-\theta} \right]
  \]
- First-order conditions for this maximization come from \( \frac{dU}{dC_{1,t}} = 0 \)
  - \( \frac{C_{2,t+1}}{C_{1,t}} = \left( \frac{1 + r_{t+1}}{1 + \rho} \right)^{\frac{1}{\theta}} \)
  - Budget constraint: \( C_{2,t+1} = (1 + r_{t+1}) ( w_t A_t - C_{1,t} ) \)
  - Note similarity of first condition to Euler equation in Ramsey model:
    - \( C \) increases or decreases over time as \( r > \rho \) or \( r < \rho \)
    - Sensitivity of consumption path to \( r \) depends on \( 1/\theta \)
- Plugging the budget constraint back into the other first-order condition yields

\[ C_{1,t} = \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t \]

o Note that numerator and denominator terms are always positive and that the denominator is always larger

o If \( \rho \approx r \) then we consume about \( \frac{1}{2} \) of income in first period and \( \frac{1}{2} \) in second, which is consistent with basic consumption smoothing

- We can show that

\[ s'(r_{t+1}) = \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} > 0 \text{ iff } \theta < 1 \]

o Change in \( r \) has income and substitution effects
  - Reward to saving is higher if \( r \uparrow \)
  - Don’t need to save as much for retirement if \( r \uparrow \)

o Remember that \( 1/\theta \) is elasticity of intertemporal substitution
  - If \( 1/\theta \) is large, then substitution effect is strong and \( s' > 0 \)
  - If \( 1/\theta \) is small, then income effect dominates and \( s' < 0 \)

o Intermediate case \( \theta = 1 \) has \( s' = 0 \) and saving rate does not depend on interest rate
  - Recall that the CRRA utility function approaches \( u = \ln(c) \) as \( \theta \to 1 \).
  - If \( \theta = 1 \), then \( s = \frac{1}{2+\rho} = \text{constant} \)

- We shall use log utility as a special case because it is simple
- What does this mean in terms of indifference curves?

Income and substitution effects cancel out as \( C_{1,t} \) is the same after increase in \( r_{t+1} \) steepens the budget constraint.
Analysis of the Diamond Model

Dynamics

- The basic equation of motion of this model is $K_{t+1} = s(r_{t+1})L_tA_tw_t$
- We want to translate this into $k_{t+1}$:
  
  $$k_{t+1} = \frac{K_{t+1}}{A_{t+1}L_{t+1}} = s(r_{t+1})\frac{A_tw_tL_t}{A_{t+1}L_{t+1}}$$
  
  $$= s(r_{t+1})\frac{w_t}{L_{t+1}}\frac{A_{t+1}}{A_t} = s[f'(k_{t+1})]\frac{f(k_t) - k_t f'(k_t)}{(1+n)(1+g)}$$

  - This equation gives $k_{t+1}$ implicitly as a function of $k_t$, but it can’t be solved in the general case.

- **Steady-state condition**
  
  - What would correspond to $\dot{k} = 0$?
  
  - $\Delta k_{t+1} = k_{t+1} - k_t = 0$ would be the equivalent in discrete time
  
  - Setting $\Delta k_{t+1} = 0$ (or $k_{t+1} = k_t = k^*$) gives the steady-state condition:
    
    $$k^* = \frac{s[f'(k^*)][f(k^*) - k^* f'(k^*)]}{(1+n)(1+g)}$$

    - This equation is difficult to work with because we don’t know the form of $f$ and we don’t even know the sign of $s'$.
    
    - Depending on the forms of $s$ and $f$, the function on the right can have a variety of shapes.

- **Special case: $\theta = 1$ (log utility) and Cobb-Douglas production function $y = k^\alpha$**

  - In this case, $s(r) = \frac{1}{2+p}$, $f'(k) = \alpha k^{\alpha-1}$, $w = (1-\alpha)k^\alpha$.

  - $k_{t+1} = \frac{(1-\alpha)k_t^\alpha}{(2+p)(1+n)(1+g)} = Dk_t^\alpha$ with positive constant $D$

  - In this case, we can graph $k_{t+1}$ as a function of $k_t$ and know its basic shape:
In this case, if we start at \(k_0\), we will converge to \(k^* = D^{1/\alpha}\) by the "cobweb" path shown.

Note effects of parameters:
- \(\rho \uparrow \Rightarrow D \downarrow\)
- \(n \uparrow \Rightarrow D \downarrow\)
- \(g \uparrow \Rightarrow D \downarrow\)
- Effects are similar to Ramsey (and Solow) model

Properties of Diamond Steady State
- In steady-state equilibrium
  - \(k\) and \(y\) are stable
  - \(Y/L\) grows at \(g\)
  - \(Y, K\) grow at \(n + g\)
- Speed of convergence:
  - \(k_{t+1} - k_t \approx \alpha (k_t - k^*)\)
  - \(\alpha \approx 1/3\), so economy moves 1/3 of way to equilibrium in each “period”
    - Note that “period” is half a lifetime, so this is not so different from Solow/Ramsey result

Diamond Model General Case?
- If we abandon the comfortable home of log utility and Cobb-Douglas we admit to strange possibilities:
Analysis of the Diamond Model

- Can have multiple equilibria

Dynamic Inefficiency in Diamond Model

- Equilibrium in Ramsey model was Pareto efficient
- Diamond model admits the possibility of inefficiency
  - It is possible that $k^*$ is greater than Golden Rule $k$
Why? Because survival in old age depends on saving lots of income regardless of rate of return.

It is possible that the saving rate that is optimal for an individual might be higher than the saving rate that leads to Golden Rule level of $k^*$

- Suppose that $k^* > k_{GR}$
  - Suppose that there was another (than capital) way of transferring money from youth to old age (Social Security) so that saving could go down
  - Young would be better off because they could consume more
  - Future generations would be better off because $k^* \downarrow$ means higher steady-state $c^*$
  - Everyone is made better off and no one worse off, so original equilibrium must not have been Pareto optimal

- How can model be inefficient? Where is the market failure?
  - No externalities, but an absent market: no way for current generation to trade effectively with future generations
  - Only way to provide for retirement is through saving, even if rate of return is zero or negative
  - From social standpoint, it is desirable to avoid low- or negative-return investment, but for individual facing retirement this is only choice
  - If benevolent government were to establish transfer scheme from young to old (like Social Security), then they would not need to accumulate useless capital in order to eat in retirement
  - You will do a problem this week looking at the effect of alternative Social Security regimes in the Diamond model

- Is this empirically relevant?
  - $k > k_{GR}$ means that $f'(k) - \delta < g + n$
  - Are interest rates lower than the GDP growth rate?
    - Probably not in U.S. steady state