[Note: This problem follows Section 5.2 of the instructor's time-series chapters. Even though the solution is there, I still want you to work out the details for yourself, using the chapter as guidance when it is useful.]

Suppose that we have a two-variable dynamic system involving two stationary variables x and y. There is contemporaneous causality running from x to y, but not vice versa. The structural model is, therefore, given by

$$x_t = \alpha_0 + \alpha_1 x_{t-1} + \theta_1 y_{t-1} + \varepsilon_t^x$$
$$y_t = \phi_0 + \phi_1 y_{t-1} + \delta_0 x_t + \delta_1 x_{t-1} + \varepsilon_t^y$$

where the ε error terms are (homoskedastic) white noise and $\mathrm{var}\left(\varepsilon_t^x\right) = \sigma_x^2$, $\mathrm{var}\left(\varepsilon_t^y\right) = \sigma_y^2$, and $\mathrm{cov}\left(\varepsilon_t^x, \varepsilon_t^y\right) = 0$. These ε terms are the "pure shocks" to x and y that are unrelated to anything in the past or anything having to do with the other variable.

1. Show that the solution of this system of equations is a VAR system that can be written

$$x_t = \beta_{x,0} + \beta_{x,1} x_{t-1} + \gamma_{x,1} y_{t-1} + v_t^x$$
$$y_t = \beta_{y,0} + \beta_{y,1} x_{t-1} + \gamma_{y,1} y_{t-1} + v_t^y.$$

What are the β and γ coefficients in terms of the α, θ, ϕ, and δ parameters?

2. Calculate the VAR error terms v in terms of the structural shocks ε. What is the variance of each of the v error terms and what is the covariance between them, expressed in terms of σ_x^2, σ_y^2, and the parameters of the model?

3. Suppose that we estimate the VAR and use the residuals $\hat{v}_t^x$ and $\hat{v}_t^y$ to calculate $\mathrm{var}\left(v_t^x\right)$, $\mathrm{var}\left(v_t^y\right)$, and $\mathrm{cov}\left(v_t^x, v_t^y\right)$. Show how we can identify σ_x^2, σ_y^2, and δ_0 from these three parameters, and that given these identifications we can identify all of the α, θ, ϕ, and δ in the structural system from the reduced-form coefficients.

4. Without doing too much actual calculation, why would it be impossible to identify the parameters of the model if y_t appeared in the equation for x_t? Why would it be impossible to identify the parameters if $\mathrm{cov}\left(\varepsilon_t^x, \varepsilon_t^y\right) = \sigma_{xy} \neq 0$?