Math 112
Numbers:
From $2 \cdot 0 = 0$ to $e^{2\pi i} = 1$

R. Mayer

Spring 2006

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"Then, my dear Glaucon, it is proper to lay down that study by law, and to persuade those who are to share in the highest things in the city to go for and tackle the art of calculation, and not as amateurs; they must keep hold of it until they are led to contemplate the very nature of numbers by thought alone, practising it not for the purpose of buying and selling like merchants or hucksters, but for war, and for the soul itself, to make easier the change from the world of becoming to real being and truth."[38, Republic 525c]

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Contents

Acknowledgements **vi**

Introduction **1**

1 Notation, Undefined Concepts and Examples **5**
1.1 Sets 5
1.2 Propositions 7
1.3 Equality 12
1.4 More Sets 14
1.5 Functions 16
1.6 *Russell's Paradox 17

2 Fields **19**
2.1 Binary Operations 19
2.2 Some Examples 25
2.3 The Field Axioms 29
2.4 Some Consequences of the Field Axioms. 33
2.5 Subtraction and Division 39
2.6 Ordered Fields 43
2.7 Absolute Value 48

3 Induction and Integers **55**
3.1 Natural Numbers and Induction 55
3.2 Integers and Rationals. 64
3.3 Recursive Definitions. 70
3.4 Summation. 75
3.5 Maximum Function 80

4 The Complexification of a Field. 83
4.1 Construction of $\mathbf{C}_F$. 83
4.2 Complex Conjugate. 89

5 Real Numbers 92
5.1 Sequences and Search Sequences 92
5.2 Completeness 97
5.3 Existence of Roots 100

6 The Complex Numbers 106
6.1 Absolute Value and Complex Conjugate 106
6.2 Geometrical Representation 108
6.3 Roots of Complex Numbers 113
6.4 Square Roots 115
6.5 Complex Functions 117

7 Complex Sequences 125
7.1 Some Examples. 125
7.2 Convergence 127
7.3 Null Sequences 129
7.4 Sums and Products of Null Sequences 134
7.5 Theorems About Convergent Sequences 136
7.6 Geometric Series 144
7.7 The Translation Theorem 148
7.8 Bounded Monotonic Sequences 153

8 Continuity 159
8.1 Compositions with Sequences 159
8.2 Continuity 161
8.3 Limits 166

9 Properties of Continuous Functions 176
9.1 Extreme Values 176
9.2 Intermediate Value Theorem 179

10 The Derivative 182
10.1 Derivatives of Complex Functions 182
10.2 Differentiable Functions on $\mathbf{R}$ 187
10.3 Trigonometric Functions 196

11 Infinite Series **202**
11.1 Infinite Series 202
11.2 Convergence Tests 206
11.3 Alternating Series 216
11.4 Absolute Convergence 221

12 Power Series **226**
12.1 Definition and Examples 226
12.2 Radius of Convergence 231
12.3 Differentiation of Power Series 233
12.4 The Exponential Function 236
12.5 Logarithms 240
12.6 Trigonometric Functions 242
12.7 Special Values of Trigonometric Functions 250
12.8 Proof of the Differentiation Theorem 253
12.9 Some XVIII-th Century Calculations 257

Bibliography **261**

A Associativity and Distributivity of Operations in $\mathbf{Z}_n$ **266**

B Hints and Answers **269**

C List of Symbols **274**

Index **278**

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