Chapter 1

Notation, Undefined Concepts and Examples

The ideas discussed in this chapter (e.g. *set*, *proposition*, *function*) are so basic that I cannot define them in terms of simpler ideas. Logically they are undefined concepts, even though I give definitions for them. My "definitions" use undefined words (e.g. *collection*, *statement*, *rule*) that are essentially equivalent to what I attempt to define. The purpose of these "definitions" and examples is to illustrate how the ideas will be used in the later chapters. I make frequent use of the undefined terms "true", "false", and "there is". It might be appropriate to spend some time discussing various opinions about the meaning of "truth" and "existence" in mathematics, but such a discussion would be more philosophical than mathematical, and would not be very relevant to anything that follows. If such questions interest you, you might enjoy reading *Philosophy of Mathematics* by Benacerraf and Putnam [7] or the article *Schizophrenia in Contemporary Mathematics* by Errett Bishop [10, pp 1-10]

Some of the terms and notation used in the examples in this chapter will be defined more precisely later in the notes. In this chapter I will assume familiar properties of numbers that you have used for many years.

1.1 Sets

1.1 Definition (Set.) A *set* is a collection of objects. A small set is often described by listing the objects it contains inside curly brackets, e.g.,

$$\{1,3,5,7,9\}$$

denotes the set of positive odd integers smaller than ten.

1.2 Notation ($\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \emptyset$.) A few sets appear so frequently that they have standard names:

$$\begin{aligned}
\mathbf{N} &= \text{set of natural numbers } = \{0,1,2,3,\cdots\}. \\
\mathbf{Z} &= \text{set of integers } = \{0,1,-1,2,-2,3,-3,\cdots\}. \\
\mathbf{Q} &= \text{set of rational numbers.} \\
&= \text{set of fractions } \frac{p}{q} \text{ where } p,q \text{ are integers and } q \neq 0. \\
\mathbf{Q}^{+} &= \text{set of positive rational numbers.} \\
\emptyset &= \text{empty set } = \{\ \} = \text{ set containing no objects.}
\end{aligned}$$

1.3 Notation ($\in, \notin$.) If A is a set and a is an object, we write

$$a \in A$$

(read this as "a is in A") to mean that a is an object in A, and we write

$$a \notin A$$

(read this as "a is not in A") to mean that a is not in A.

1.4 Example. Thus we have

$$\begin{aligned}
2 &\in \mathbf{Z}, \\
-2 &\notin \mathbf{N}, \\
2 &\in \{1,2,5\} \\
\{1,2\} &\notin \{1,2,5\}. \qquad (1.5)
\end{aligned}$$

To see why (1.5) is true, observe that the only objects in $\{1,2,5\}$ are 1, 2, and 5. Since

$$\{1,2\} \neq 1 \text{ and } \{1,2\} \neq 2 \text{ and } \{1,2\} \neq 5$$

it follows that $\{1,2\} \notin \{1,2,5\}$.

1.6 Definition (Subset, $\subset$.) Let A and B be sets. We say that A *is a subset of* B and write

$$A \subset B$$

if and only if every object in A is also in B.

1.7 Example.

$$\begin{aligned} \mathbf{N} &\subset \mathbf{Z}, \\ \emptyset &\subset \mathbf{Z}, \\ \mathbf{Z} &\subset \mathbf{Z}, \\ \{1,2\} &\subset \{1,2,3\}, \\ \{1\} &\subset \mathbf{Z}, \end{aligned}$$

are all true statements. However

$$1 \subset \mathbf{Z} \tag{1.8}$$

is not a statement, but an ungrammatical phrase, since $A \subset B$ has only been defined when A and B are sets, and 1 is not a set.

1.9 Definition (Set equality.) Two sets A and B are considered to be the same if and only if they contain exactly the same objects. In this case we write

$$A = B.$$

Thus $A = B$ if and only if $A \subset B$ and $B \subset A$.

1.10 Example.

$$\begin{aligned} \{1,2,3\} &= \{3,1,1,2\} \\ \{1,2,3,4\} &= \{1,2+1,3+1,1+1,2+2\} \end{aligned}$$

1.2 Propositions

1.11 Definition (Proposition.) A *proposition* is a statement that is either true or false.

1.12 Example. Both

$$1 + 1 = 2$$

and

$$1 + 1 = 3$$

are propositions. The first is true and the second is false. I will consider

13 is a prime number

to be a proposition, because I expect that you know what a prime number is. However, I will not consider

13 is an unlucky number

to be a proposition (unless I provide you with a definition for *unlucky number*).

The proposition

$$\{1\} \subset \mathbf{N}$$

is true, and the proposition

$$\mathbf{N} \in \mathbf{N}$$

is false, but

$$1 \subset \mathbf{N}$$

is not a proposition but rather a meaningless statement (cf (1.8)). Observe that "$x \subset y$" makes sense whenever x and y are sets, and " $x \in y$" makes sense when y is a set, and x is any object. Similarly

$$\frac{1}{0} = 5$$

is meaningless rather than false, since division by zero is not defined., i.e. I do not consider $\frac{1}{0}$ to be a name for any object.

1.13 Definition (and, or, not.) If P and Q are propositions, then

$$P \text{ or } Q \qquad P \text{ and } Q \qquad \text{not } P$$

are propositions, and (P or Q) is true if and only if at least one of P, Q is true; (P and Q) is true if and only if both of P, Q are true; (not P) is true if and only if P is false.

1.14 Example.

$$\begin{aligned}(1+1=2) &\quad \text{and} \quad (2+2=4),\\ (1+1=2) &\quad \text{or} \quad (1+1=3),\\ (1+1=2) &\quad \text{or} \quad (2+2=4),\end{aligned}$$

are all true propositions.

1.15 Notation ($\neq, \notin$.) We abbreviate

$$\text{not } (a=b) \text{ by } a \neq b,$$

and we abbreviate

$$\text{not } (a \in A) \text{ by } a \notin A.$$

1.16 Notation ($\Longrightarrow$) If P and Q are propositions, we write

$$P \Longrightarrow Q \tag{1.17}$$

to denote the proposition "P implies Q".

1.18 Example. If x, y, z are integers then all of the following are true:

$$(x=y) \quad \Longrightarrow \quad (z \cdot x = z \cdot y). \tag{1.19}$$

$$(x=y) \quad \Longrightarrow \quad (x+z = y+z). \tag{1.20}$$

$$(x \neq y) \quad \Longrightarrow \quad ((z=x) \Longrightarrow (z \neq y)). \tag{1.21}$$

The three main properties of implication that we will use are:

If P is true, and $(P \Longrightarrow Q)$ is true, then Q is true.
If $(P \Longrightarrow Q)$ is true and Q is false, then P is false.
If $(P \Longrightarrow Q)$ is true and $(Q \Longrightarrow R)$ is true, then $(P \Longrightarrow R)$ is true.(1.22)

We denote property (1.22) by saying that $\Longrightarrow$ is *transitive.*

1.23 Example. The meaning of a statement like

$$(1=2) \Longrightarrow (5=7) \tag{1.24}$$

or

$$(1=2) \Longrightarrow (5 \neq 7) \tag{1.25}$$

may not be obvious. I claim that both (1.24) and (1.25) should be true.
“Proof” of (1.24):

$$\begin{aligned}(1=2) &\implies (2\cdot 1 = 2\cdot 2) \text{ (by (1.19))},\\ (2\cdot 1 = 2\cdot 2) &\implies (2\cdot 1+3 = 2\cdot 2+3) \text{ (by (1.20))},\end{aligned}$$

and

$$(2\cdot 1+3 = 2\cdot 2+3) \implies (5=7),$$

so by transitivity of $\implies$,

$$(1=2) \implies (5=7). \parallel$$

“Proof” of (1.25):

$$(1=2) \implies (1+4=2+4) \text{ (by (1.20))}$$

so

$$\begin{aligned}(1=2) &\implies (5=6),\\ (5=6) &\implies (5\neq 7) \text{ (by (1.21), since } 6\neq 7),\end{aligned}$$

so

$$(1=2) \implies (5\neq 7) \text{ by transitivity of } \implies. \parallel$$

The previous example is supposed to motivate the following assumption:

A false proposition implies everything,

i.e.

If P is false, then $(P \implies Q)$ is true for all propositions Q.

1.26 Example. For every $x \in \mathbf{Z}$, the proposition

$$x = 2 \implies x^2 = 4$$

is true. Hence all three of the statements below are true:

$$2 = 2 \implies 2^2 = 4, \tag{1.27}$$

$$-2 = 2 \implies (-2)^2 = 4, \tag{1.28}$$

$$3 = 2 \implies 3^2 = 4. \tag{1.29}$$

Proposition (1.28) is an example of a false statement implying a true one, and proposition (1.29) is an example of a false statement implying a false one. Equations (1.27) and (1.28) together provide motivation for the assumption.

Every statement implies a true statement;

i.e.

If Q is true then $(P \implies Q)$ is true for all propositions P.

The following table shows the conditions under which $(P \implies Q)$ is true.

P	Q	$P \implies Q$
true	true	true
true	false	false
false	true	true
false	false	true

Thus a true statement does not imply a false one. All other sorts of implications are valid.

1.30 Notation ($P \implies Q \implies R \implies S$.) Let P, Q, R ,S be propositions. Then

$$P \implies Q \implies R \implies S \tag{1.31}$$

is an abbreviation for

$$((P \implies Q) \text{ and } (Q \implies R)) \text{ and } (R \implies S).$$

It follows from transitivity of $\implies$ that if (1.31) is true, then $P \implies S$ is true.

Note that (1.31) is *not* an abbreviation for

$$((P \text{ and } (P \implies Q)) \text{ and } (Q \implies R)) \text{ and } (R \implies S);$$

i.e., when I write (1.31), I do not assume that P is true.

1.32 Definition (Equivalence of propositions, $\iff$.) Let P, Q be propositions. We say that P and Q are *equivalent* and write

$$P \iff Q$$

(read this "P is equivalent to Q" or "P if and only if Q") to mean

$$(P \implies Q) \text{ and } (Q \implies P) \tag{1.33}$$

If either (P, Q are both true) or (P, Q are both false), then $(P \iff Q)$ is true. If one of P, Q is false, and the other is true, then one of $(P \implies Q)$, $(Q \implies P)$ has the form $\langle\text{true}\rangle \implies \langle\text{false}\rangle$, and hence in this case $(P \iff Q)$ is false. Thus

$$(P \iff Q) \text{ is true if and only if } P, Q \text{ are both true or both false.}$$

1.3 Equality

1.34 Notation (=.) Let x and y be (names of) objects. I write

$$x = y$$

to mean that x and y are names for the same object. I will not make a distinction between an object and its name.

$$\text{For all objects } x, \quad x = x. \tag{1.35}$$

We describe this property by saying that *equality is reflexive.*

$$\text{For all objects } x, y, \quad (x = y) \implies (y = x). \tag{1.36}$$

We describe this property by saying that *equality is symmetric.*

$$\text{For all objects } x, y, z \quad ((x = y) \text{ and } (y = z)) \implies (x = z). \tag{1.37}$$

We describe this property by saying that *equality is transitive.*

Let P be a proposition involving the object x. Let Q be a proposition obtained by replacing any or all occurrences of x in P by y. Then $P \iff Q$. We call this property of equality the *substitution property.*

1.38 Examples. Suppose that x, y are integers, and $x = y$. Then

$$\big((x + 3)(x + 4) = 28 + x\big) \iff \big((x + 3)(y + 4) = 28 + y\big),$$

and

$$\big((x + 3)(x + 4) = 28 + x\big) \iff \big((y + 3)(y + 4) = 28 + y\big),$$

and

$$\big((x+3)(x+4)=28+x\big) \iff \big((y+3)(x+4)=28+x\big).$$

We will frequently make statements like

$$(x=y) \implies (x+3=y+3).$$

The justification for this is

$$x+3=x+3 \qquad \text{(by reflexivity of } = .)$$

Hence, if $x=y$, then by the substitution property,

$$x+3=y+3.$$

1.39 Warning. Because we are using a vague notion of proposition, the substitution property of equality as stated is not precisely true. For example, although

$$5=2+3 \tag{1.40}$$

and

$$5\cdot 4=20 \tag{1.41}$$

are both true, the result of substituting the 5 in the second equation by $2+3$ yields

$$2+3\cdot 4=20$$

which is false.

The proper conclusion that follows from (1.40) and (1.41) is

$$(2+3)\cdot 4=20.$$

(The use of parentheses is discussed in Remark 2.50.)

1.42 Notation ($a=b=c=d$.) Let a,b,c,d be objects. We write

$$a=b=c=d \tag{1.43}$$

as an abbreviation for

$$((a=b) \text{ and } (b=c)) \text{ and } (c=d).$$

If (1.43) is true, then by several applications of transitivity, we conclude that

$$a=d.$$

1.4 More Sets

1.44 Definition (Proposition Form.) Let S be a set. A *proposition form* P on S is a rule that assigns to each element x of S a unique proposition, denoted by $P(x)$.

1.45 Examples. Let

$$P(n) = \text{“}n^2 - 6n + 8 = 0\text{” for all } n \in \mathbf{Z}.$$

Then P is a proposition form on $\mathbf{Z}$. $P(0)$ is false, and $P(2)$ is true. Note that P is neither true nor false. A proposition form is not a proposition.

Let

$$Q(n) = \text{“}n^2 - 4 = (n-2)(n+2)\text{” for all } n \in \mathbf{Z}. \tag{1.46}$$

Then Q is a propostion form, and $Q(n)$ is true for all $n \in \mathbf{Z}$. Note that Q is not a proposition, but if

$$R = \text{“}n^2 - 4 = (n-2)(n+2) \text{ for all } n \in \mathbf{Z}\text{”} \tag{1.47}$$

then R is a proposition and R is true. Make sure that you see the difference between the right sides of (1.46) and (1.47). The placement of the quotation marks is crucial. When I define a proposition I will often enclose it in quotation marks, to prevent ambiguity. Without the quotation marks, I would not be able to distinguish between the right sides of (1.46) and (1.47). If I see a statement like

$$P(n) = n^2 - 6n + 8 = 0$$

without quotation marks, I immediately think this is a statement of the form $x = y = z$ and conclude that $P(n) = 0$.

1.48 Notation. Let S be a set, and let P be a proposition form on S. Then

$$\{x \in S\colon P(x)\} \tag{1.49}$$

denotes the set of all objects x in S such that $P(x)$ is true. (Read (1.49) as “the set of all x in S such that $P(x)$”.)

1.50 Examples.

$$\{x \in \mathbf{N}\colon x^2 - 6x + 8 = 0\} = \{2, 4\}$$

$$\{x \in \mathbf{Z}\colon x = 2n \text{ for some } n \in \mathbf{Z}\} = \text{ set of even integers.}$$

Variations on this notation are common. For example,

$$\{n^2 + n \colon n \in \mathbf{Z}\}$$

represents the set of all numbers of the form $n^2 + n$ where $n \in \mathbf{Z}$.

1.51 Definition (Union, intersection, difference.) Let A be a set, let $\mathcal{S}$ be the set of all subsets of A, and let R, T be elements of $\mathcal{S}$. We define the *intersection* $R \cap T$ of R and T by

$$R \cap T = \{x \in A \colon x \in R \text{ and } x \in T\};$$

we define the *union* $R \cup T$ of R and T by

$$R \cup T = \{x \in A \colon x \in R \text{ or } x \in T\};$$

and we define the difference $R \setminus T$ by

$$R \setminus T = \{x \in R \colon x \notin T\}.$$

1.52 Examples. If $R = \{1, 2, 3\}$ and $T = \{2, 3, 4, 5\}$, then

$$\begin{aligned} R \cap T &= \{2, 3\} \\ R \cup T &= \{1, 2, 3, 4, 5\} \\ R \setminus T &= \{1\} \\ T \setminus R &= \{4, 5\}. \end{aligned}$$

1.53 Definition (Ordered pairs and triples.) Let a, b, c be objects (not necessarily all different). The ordered pair (a, b) is a set-like combination of a and b into a single object, in which a is designated as the *first element* and b is designated as the *second element*. The ordered triple (a, b, c) is a similar construction having a for its first element, b for its second element and c for its third element. Two ordered pairs (triples) are equal if and only if they have the same first elements, the same second elements, (and the same third elements). Thus

$$\begin{aligned} (a, b) = (b, a) &\iff b = a. \\ (a, b) = (x, y) &\iff a = x \text{ and } b = y. \\ (a, b, c) = (x, y, z) &\iff (a = x) \text{ and } (b = y) \text{ and } (c = z). \end{aligned}$$

1.54 Warning. Ordered pairs should not be confused with sets.

$$\begin{aligned} \{1,2\} &= \{2,1\}. \\ (1,2) &\neq (2,1). \end{aligned}$$

1.55 Definition (Cartesian product, $\times$.) If A and B are sets, we define the set $A \times B$ by

$$\begin{aligned} A \times B &= \text{ the set of all ordered pairs } (a,b) \text{ where } a \in A \text{ and } b \in B. \\ A^2 &= A \times A. \\ A^3 &= \text{ the set of all ordered triples } (a,b,c) \text{ where } a,b,c \text{ are in } A. \end{aligned}$$

$A \times B$ is called the *Cartesian Product* of A and B.

1.56 Example. If $\mathbf{R}$ is the set of real numbers, then $\mathbf{R}^2$ is the set of all ordered pairs of real numbers. You are familiar with the fact that ordered pairs of real numbers can be represented as points in the plane, so you can think of $\mathbf{R}^2$ or $\mathbf{R} \times \mathbf{R}$ as being the points in the plane.

1.5 Functions

1.57 Definition (Function.) Let A, B be sets, and let f be a rule that assigns to each element a in A a unique element (denoted by $f(a)$) in B. The ordered triple (A, B, f) is called a *function with domain A and codomain B*. We write

$$f\colon A \to B$$

to indicate that (A, B, f) is a function. It follows from the definition that two functions are equal if and only if they have the same domain, the same codomain, and the same rule: If $f\colon A \to B$ and $g\colon A \to B$, I say that the rule f and the rule g are the same if and only if $f(a) = g(a)$ for all $a \in A$. We usually say "the function f" when we mean "the function (A, B, f)," i.e., we name a function by giving just the name for its rule.

1.58 Examples. Let

$$\begin{aligned} &f\colon \mathbf{N} \to \mathbf{Z}, \\ &g\colon \mathbf{Z} \to \mathbf{Z}, \\ &h\colon \mathbf{Z} \to \mathbf{N}, \\ &k\colon \mathbf{Z} \to \mathbf{Z}, \end{aligned}$$

be defined by

$$\begin{aligned} f(n) &= (n-2)(n-3) \text{ for all } n \in \mathbf{N}, \\ g(n) &= (n-2)(n-3) \text{ for all } n \in \mathbf{Z}, \\ h(n) &= (n-2)(n-3) \text{ for all } n \in \mathbf{Z}, \\ k(w) &= w^2 - 5w + 6 \text{ for all } w \in \mathbf{Z}. \end{aligned}$$

Then

$$\begin{aligned} f &\neq g \quad (f \text{ and } g \text{ have different domains }) \\ g &\neq h \quad (g \text{ and } h \text{ have different codomains }) \\ g &= k. \end{aligned}$$

If P is a proposition form on a set S, then P determines a function whose domain is S and whose codomain is the set of all propositions.

1.6 *Russell's Paradox

There are some logical paradoxes connected with the theory of sets. The book *The Foundations of Mathematics* by Evert Beth discusses 17 different paradoxes[9, pp. 481-492]. Here I discuss just one of these which was published by Bertrand Russell in 1903[43, ¶78,¶¶100-106].

Let $\mathcal{S}$ be the set of all sets, let $\mathcal{I}$ be the set of all infinite sets, and let $\mathcal{F}$ be the set of all finite sets. Then we have

$$\begin{array}{ccc} \mathcal{F} \in \mathcal{S} & \mathcal{F} \in \mathcal{I} & \mathcal{F} \notin \mathcal{F} \\ \mathcal{I} \in \mathcal{S} & \mathcal{I} \in \mathcal{I} & \mathcal{I} \notin \mathcal{F} \\ \mathcal{S} \in \mathcal{S} & \mathcal{S} \in \mathcal{I} & \mathcal{S} \notin \mathcal{F} \\ 2 \notin \mathcal{S} & \{2\} \notin \mathcal{I} & \{\mathcal{S}\} \in \mathcal{F} \end{array}$$

Here $\mathcal{F} \in \mathcal{I}$ since there are infinitely many finite sets. $\{\mathcal{S}\} \in \mathcal{F}$ since $\{\mathcal{S}\}$ contains just one element, which is the set of all sets. Also $2 \notin \mathcal{S}$ since 2 is not a set. Next, let

$$\mathcal{R} = \{x \in \mathcal{S} : x \notin x\}.$$

Then for all $x \in \mathcal{S}$ we have

$$x \in \mathcal{R} \iff x \notin x. \tag{1.59}$$

Thus

$$\begin{array}{rcl} \mathcal{S} \notin \mathcal{R} & \text{since} & \mathcal{S} \in \mathcal{S}, \\ \mathcal{I} \notin \mathcal{R} & \text{since} & \mathcal{I} \in \mathcal{I}, \\ \mathcal{F} \in \mathcal{R} & \text{since} & \mathcal{F} \notin \mathcal{F}, \\ \mathbf{Z} \in \mathcal{R} & \text{since} & \mathbf{Z} \notin \mathbf{Z}. \end{array}$$

We now ask whether $\mathcal{R}$ is in $\mathcal{R}$. According to (1.59),

$$\mathcal{R} \in \mathcal{R} \iff \mathcal{R} \notin \mathcal{R},$$

i.e., $\mathcal{R}$ is in $\mathcal{R}$ if and only if it isn't!

I believe that this paradox has never been satisfactorily explained. A large branch of mathematics (axiomatic set theory) has been developed to get rid of the paradox, but the axiomatic approaches seem to build a fence covered with "keep out" signs around the paradox rather than explaining it. Observe that the discussion of Russell's paradox does not involve any complicated argument: it lies right on the surface of set theory, and it might cause one to wonder what other paradoxes are lurking in a mathematics based on set theory.

1.60 Warning. Thinking too much about this sort of thing can be dangerous to your health.

> The poet and grammarian Philitas of Cos is even said to have died prematurely from exhaustion, owing to his desperate efforts to solve the paradox.[9, page 493]

Philitas was concerned about a different paradox, but Russell's paradox is probably more deadly.