Appendix B

Hints and Answers

Exercise 2.11: In each case there is only one invertible element.

Exercise 2.40: Note that the calculator sum of a very small number and a very large number is the large number. Note that the calculator product of two small positive numbers is 0.

Exercise 2.55: The system $(\mathbf{Q}, \oplus, \odot)$ fails two axioms. The other three each fail one axiom.

Exercise 2.78: Use (2.75).

Exercise 2.90: For part (b), use (2.74).

Exercise 2.93: Part e) can be done quickly by using parts a), b) and d).

Exercise 2.123: One of the conditions is that a and b have the same sign.

Exercise 2.135: There are nine cases to consider (three for a and three for y). They can be reduced to five cases, one of which is $((x = 0$ or $y = 0)$.

Exercise 2.144: There is a *very* short proof for a).

Exercise 2.145: Apply the product formula for absolute values to $|a \cdot a^{-1}|$.

Exercise 2.154: Use (2.138).

Exercise 3.24: You can take $S = F^+$.

Exercise 3.32: Suppose n is both even and odd, and derive a contradiction by using theorems 3.15 and 3.19.

Exercise 3.43: For part b) use part a) together with exercise 3.32

Exercise 3.57: You can let $P(n) =$"for all $m \in \mathbf{N}((a^n)^m = a^{(nm)})$" or $P(m) =$"for all $n \in \mathbf{N}((a^n)^m = a^{(nm)})$".

Exercise 3.82: f) Note that $x^6 + a^6 = (x^2)^3 - (-a^2)^3$.

Exercise 3.85: $S_n = \frac{n}{n+1}$.

Exercise 3.87: $T_n = n^2$.

Exercise 4.19: a) Note that $\frac{1}{z} = z^{-1}$. b) The solutions are i and $\frac{2}{5} - \frac{1}{5}i$.

Exercise 4.23: For part c), write $z = w \cdot \frac{z}{w}$ and use part b). Remark 4.22 is used for part f).

Exercise 4.25: a) 1. b) 32i.

Exercise 5.15: I let $P(n) =$"$2^n \geq n$ and $2^n \geq 1$" and I used the fact that $2^{n+1} = 2^n + 2^n$. (It would be reasonable to assume $2^n \geq 1$ for all n, but I proved it).

Exercise 5.48: Suppose $f(x) = a = f(t)$, and use trichotomy to show that $x = t$.

Exercise 5.51: a) If $x^q = y^q$, then $x = y$. b) If $x^{qr} = y^{rq}$, then $x = y$. (We know the laws of exponents for integer exponents.)

Exercise 5.54: Raise both sides of the equation to the same integer power, and use laws of exponents for integer powers.

Exercise 6.5: e) Notice that $x^2 \leq x^2 + y^2$, and use theorem 2.128.

Exercise 6.26: The roots are $\pm\left(\frac{5}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$.

Exercise 7.21: I showed $|(\frac{2+i}{3})|^2 < .7$, so that $|(\frac{2+i}{3})| < \sqrt{.7}$. Then I used the fact that $\{.7^n\}$ is known to be a null sequence, and used the root theorem and the comparison theorem.

Exercise 7.22: All three sequences are null sequences. For the last sequence, I showed $\frac{n^2+6}{n^2+3} \leq 2$ for all $n \geq 1$. For the first sequence $\{a_n\}$, I showed $|a_n| \leq \frac{10000}{\sqrt{n}}$ for all $n \geq 1$.

Exercise 7.24: The set of all complex sequences does not form a field. All of the field axioms except one hold.

Exercise 7.28: $N_{fg}(\epsilon) = \max(N_f(\epsilon), N_g(1))$ works.

Exercise 7.44: You can take $N_{fg}(\epsilon) = N_f(\frac{\epsilon}{M})$, where $|g(n)| \leq M$ for all $n \in \mathbf{N}$.

Exercise 7.49: Note that $g = (f + g) - f$, and show that the assumption $f + g$ converges is contradictory.

Exercise 7.50: Note that if $c \neq 0$, then $f = \frac{1}{c} \cdot cf$.

Exercise 7.56: Multiply numerator and denominator of $\sqrt{f(n)} - \sqrt{L}$ by $\sqrt{f(n)} + \sqrt{L}$.

Exercise 7.57: All of the sequences converge. $h \to \frac{-4i}{3}$ and $l \to \frac{1}{2}$.

Exercise 7.58: If B_1 is a bound for f, and B_2 is a bound for g, then $B_1 + B_2$ is a bound for $f + g$.

Exercise 7.60: A convergent sequence is the sum of a null sequence and a constant sequence. Now use exercise 7.58.

Exercise 7.69: $\{a_n\} \to \frac{37}{55}$.

Exercise 7.70: One of the sequences diverges. One of the limits is $\frac{255}{82} + \frac{20}{41}i$.

Exercise 7.74: Apply theorem 7.73 to $g - f$.

Exercise 7.75: The statement is false.

Exercise 7.86: The sequence converges to $\frac{A}{a}$ or to $\frac{B}{b}$ or else it diverges. (You should find exact conditions on a, b, A, and B corresponding to each case.)

Exercise 7.94: If $L \leq a_n \leq U$ for all $n \in \mathbf{N}$, then either $|L| + |U|$ or $\max(|L|, |U|)$ will be a bound for $\{a_n\}$

Exercise 7.100: Use corollary 7.96 to show that the sequence converges. Note $a_{n+1} = a_n \cdot \frac{60}{n+1}$. Use the translation theorem to show that the limit is 0.

Exercise 8.4: I used the polar decomposition for α.

Exercise 8.19: Yes. In fact every function from **N** to **C** is continuous. The only integer p that satisfies $|p| < \frac{1}{2}$ is zero.

Exercise 8.43: a) Limit does not exist. b) Let $\{z_n = x_n + iy_n\}$ be a sequence converging to 0. Apply null-times-bounded theorem to $\{f(z_n)\}$. c) I looked at sequences $\{\frac{\omega}{n}\}$, where ω is a direction.

Exercise 9.10: Four of the requested five functions exist. The other one doesn't.

Exercise 9.13: Apply theorem 9.11 to g, where $g(x) = f(x) - y$

Exercise 9.17: All four functions exist. I described k by drawing its graph.

Exercise 9.19: Assume $F(4) < 0$ and derive a contradiction from the intermediate value theorem.

Exercise 10.10: h is nowhere differentiable. The same sequences that show complex conjugation is nowhere differentiable show that h is also.

Exercise 10.17: $D_a(fg)(z) = f(a)D_a g(z) + g(a)D_a f(z) + (z-a)D_a f(z) D_a g(z)$.

Exercise 10.19: Write $f^n = \frac{1}{f^{-n}}$, and use the reciprocal rule and exercise 10.18.

Exercise 10.20: Use the product rule and the reciprocal rule.

Exercise 10.31: Only one of the three statements is true.

Exercise 10.37: Use the result of exercise 6.36

Exercise 10.50: For both parts, compare with $\{\frac{1}{n}\}_{n\geq 1}$

Exercise 10.51: The exact value of $\cos(i)$ is $\frac{1}{2}(e + \frac{1}{e})$. You can't prove this (because we have not yet defined e), but you can check your answer using this.

Exercise 11.19: c) Note that the comparison test holds only for sequences of non-negative numbers.

Exercise 11.28: In most cases there is one value for which the ratio test gives no information. However the exceptional values are usually standard test series. For part g) the exceptional value was considered in example 11.27.

Exercise 11.48: b) Show that the real and imaginary parts of the series both converge.

Exercise 11.49: Both series converge absolutely for $|z| < 1$ and diverge for $|z| \geq 1$. (I used the ratio test.)

Exercise 12.6: a) If $z = -1$ then $z^{2^n} = 1$ for all $n \geq 1$, and if $z = i$, then $z^{z^n} = 1$ for all $n \geq 2$. Except for the first few terms, both series are geometric series,

Exercise 12.14: a) Consider geometric series of the form $\sum\{c^n z^n\}$. c) Cf exercise 12.5.a

Exercise 12.20: The sums are $\frac{z}{(1-z)^2}$ and $\frac{z+z^2}{(1-z)^3}$ Get the second by differentiating the first.

Exercise 12.33: a) Use induction and (12.29). b) Use part a) and (12.28).

Exercise 12.34: Calculate $\exp(p \cdot t) = \exp(q \cdot (\frac{p}{q} \cdot t))$ using exercise 12.33

Exercise 12.42: a) Write $a = \exp(\ln(a))$ and use (12.28). b) Write $a = \exp(\ln(a))$ and use exercise 12.34.

Exercise 12.51: Write the trig functions in terms of exponentials, and use $e^a e^b = e^{a+b}$ several times.

Exercise 12.59: Everything follows from $e^{\frac{i\pi}{2}} = i$ and $e^{it} = \cos(t) + i\sin(t)$ and $e^a e^b = e^{a+b}$.

Exercise 12.70: $\cos(\frac{2\pi}{5}) = \frac{\sqrt{5}-1}{4}$.