Appendix C

Prerequisites

C.1 Properties of Real Numbers

Algebraic Laws

Commutative laws for addition and multiplication: If a and b are arbitrary real numbers then

$$\begin{aligned} a+b &= b+a, && \text{(C.1)} \\ ab &= ba. && \text{(C.2)} \end{aligned}$$

Associative laws for addition and multiplication: If a, b, and c are arbitrary real numbers then

$$\begin{aligned} (a+b)+c &= a+(b+c), && \text{(C.3)} \\ (ab)c &= a(bc). && \text{(C.4)} \end{aligned}$$

As a consequence of equations C.3 and C.4 we usually omit the parentheses in triple sums or products, and write $a + b + c$ or abc. We know that all meaningful ways of inserting parentheses yield the same result.

Distributive laws: If a, b and c are arbitrary real numbers, and d is an arbitrary non-zero real number then

$$\begin{aligned} c(a+b) &= ca+cb, && \text{(C.5)} \\ c(a-b) &= ca-cb, && \text{(C.6)} \\ (a+b)c &= ac+bc, && \text{(C.7)} \end{aligned}$$

$$
\begin{aligned}
(a-b)c &= ac-bc, && \text{(C.8)}\\
(a+b)/d &= a/d+b/d, && \text{(C.9)}\\
(a-b)/d &= a/d-b/d. && \text{(C.10)}
\end{aligned}
$$

Properties of zero and one: The rational numbers 0 and 1, have the property that for all real numbers a

$$
\begin{aligned}
a+0 &= a, && \text{(C.11)}\\
0+a &= a, && \text{(C.12)}\\
a\cdot 1 &= a, && \text{(C.13)}\\
1\cdot a &= a, && \text{(C.14)}\\
0\cdot a &= 0, && \text{(C.15)}\\
a\cdot 0 &= 0. && \text{(C.16)}
\end{aligned}
$$

Moreover

$$0 \neq 1, \tag{C.17}$$

and

$$\text{if } \ ab=0 \text{ then } \ a=0 \text{ or } \ b=0 \text{ (or both).} \tag{C.18}$$

Additive and multiplicative inverses: For each real number a there is a real number $-a$ (called the *additive inverse of* a) and for each non-zero real number b there is a real number b^{-1} (called *the multiplicative inverse of* b) such that

$$
\begin{aligned}
a+(-a) &= 0, && \text{(C.19)}\\
(-a)+a &= 0, && \text{(C.20)}\\
b\cdot b^{-1} &= 1, && \text{(C.21)}\\
b^{-1}\cdot b &= 1, && \text{(C.22)}\\
-0 &= 0, && \text{(C.23)}\\
1^{-1} &= 1. && \text{(C.24)}
\end{aligned}
$$

Moreover for all real numbers a, c and all non-zero real numbers b

$$
\begin{aligned}
-(-a) &= a, && \text{(C.25)}\\
a-c &= a+(-c), && \text{(C.26)}\\
a/b &= a\cdot b^{-1}, && \text{(C.27)}
\end{aligned}
$$

$$\begin{aligned}
b^{-1} &= 1/b, &\text{(C.28)}\\
(ab)^{-1} &= a^{-1}b^{-1} &\text{(C.29)}\\
-a &= (-1)\cdot a, &\text{(C.30)}\\
(b^{-1})^{-1} &= b, &\text{(C.31)}\\
(-a)(-c) &= ac, &\text{(C.32)}\\
(-a)c &= a(-c) = -(ac), &\text{(C.33)}\\
-(\frac{a}{b}) &= \frac{-a}{b} = \frac{a}{-b}. &\text{(C.34)}
\end{aligned}$$

Note that by equation C.33, the expression $-xy$ without parentheses is unambiguous, i.e. no matter how parentheses are put in the result remains the same.

Cancellation laws: Let a, b, c be real numbers. Then

$$\begin{aligned}
&\text{if } a+b=a+c, &&\text{then} && b=c. &\text{(C.35)}\\
&\text{if } b+a=c+a, &&\text{then} && b=c. &\text{(C.36)}\\
&\text{if } ab=ac \text{ and } a\neq 0 &&\text{then} && b=c. &\text{(C.37)}\\
&\text{if } ba=ca \text{ and } a\neq 0 &&\text{then} && b=c. &\text{(C.38)}
\end{aligned}$$

Some miscellaneous identities: For all real numbers a, b, c, d, x

$$\begin{aligned}
a^2-b^2 &= (a-b)(a+b), &\text{(C.39)}\\
(a+b)^2 &= a^2+2ab+b^2, &\text{(C.40)}\\
(a-b)^2 &= a^2-2ab+b^2, &\text{(C.41)}\\
(x+a)(x+b) &= x^2+(a+b)x+ab, &\text{(C.42)}\\
(a+b)(c+d) &= ac+ad+bc+db. &\text{(C.43)}
\end{aligned}$$

Moreover, if $b \neq 0$ and $d \neq 0$ then

$$\begin{aligned}
\frac{a}{b}\cdot\frac{c}{d} &= \frac{ac}{bd}. &\text{(C.44)}\\
\frac{a}{b}+\frac{c}{d} &= \frac{ad+bc}{bd}. &\text{(C.45)}\\
\frac{a}{b}-\frac{c}{d} &= \frac{ad-bc}{bd}. &\text{(C.46)}
\end{aligned}$$

If w, x, y, z are real numbers, then

$$w-x-y+z+y \text{ means } w+((-x)+((-y)+(z+y))) \tag{C.47}$$

i.e. the terms of the sum are associated from right to left. It is in fact true that all meaningful ways of introducing parentheses into a long sum yield the same result, and we will assume this. I will often make statements like

$$w - x - y + z + y = w + z - x \tag{C.48}$$

without explanation. Equation C.48 can be proved from our assumptions, as is shown below, but we will usually take such results for granted.

Proof of equation C.48. Let w, x, y, z be real numbers. Then

$$\begin{aligned}
w - x - y + z + y &= w + ((-x) + ((-y) + (z + y))) \text{ by C.47} \\
&= w + ((-x) + ((-y) + (y + z))) \text{ by C.1} \\
&= w + ((-x) + (((-y) + y) + z)) \text{ by C.3} \\
&= w + ((-x) + (0 + z)) \text{ by C.20} \\
&= w + ((-x) + z) \text{ by C.12} \\
&= w + (z + (-x)) \text{ by C.1} \\
&= w + z - x \text{ by C.47.}
\end{aligned}$$

Order Laws

There is a relation $<$ (*less than*) defined on the real numbers such that for each pair a, b of real numbers, the statement "$a < b$" is either true or false, and such that the following conditions are satisfied:

Trichotomy law: For each pair a, b of real numbers exactly one of the following statements is true:

$$a < b, \qquad a = b, \qquad b < a. \tag{C.49}$$

We say that a real number p is *positive* if and only if $p > 0$, and we say that a real number n is *negative* if and only if $n < 0$. Thus as a special case of the trichotomy law we have:

If a is a real number, then exactly one of the following statements is true:

$$a \text{ is positive}, \qquad a = 0, \qquad a \text{ is negative}. \tag{C.50}$$

Sign laws: Let a, b be real numbers. Then

$$\text{if } a > 0 \text{ and } b > 0 \quad \text{then} \quad ab > 0 \text{ and } a/b > 0, \tag{C.51}$$

$$\text{if } a < 0 \text{ and } b > 0 \quad \text{then} \quad ab < 0 \text{ and } a/b < 0, \tag{C.52}$$
$$\text{if } a > 0 \text{ and } b < 0 \quad \text{then} \quad ab < 0 \text{ and } a/b < 0, \tag{C.53}$$
$$\text{if } a < 0 \text{ and } b < 0 \quad \text{then} \quad ab > 0 \text{ and } a/b > 0, \tag{C.54}$$
$$\text{if } a > 0 \text{ and } b > 0 \quad \text{then} \quad a + b > 0, \tag{C.55}$$
$$\text{if } a < 0 \text{ and } b < 0 \quad \text{then} \quad a + b < 0. \tag{C.56}$$

Also,

$$a \text{ is positive if and only if } -a \text{ is negative,} \tag{C.57}$$

and

$$a \text{ is positive if and only if } a^{-1} \text{ is positive.} \tag{C.58}$$

$$\text{if } ab > 0 \text{ then either } (a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0). \tag{C.59}$$

$$\text{if } ab < 0 \text{ then either } (a > 0 \text{ and } b < 0) \text{ or } (a < 0 \text{ and } b > 0). \tag{C.60}$$

$$\text{if } a/b > 0 \text{ then either } (a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0). \tag{C.61}$$

$$\text{if } a/b < 0 \text{ then either } (a > 0 \text{ and } b < 0) \text{ or } (a < 0 \text{ and } b > 0). \tag{C.62}$$

It follows immediately from the sign laws that for all real numbers a

$$a^2 \geq 0 \text{ and } \quad \text{if } a \neq 0 \text{ then } a^2 > 0. \tag{C.63}$$

Here, as usual a^2 means $a \cdot a$.

Transitivity of $<$: Let a, b, c be real numbers. Then

$$\text{if } a < b \text{ and } b < c \text{ then } a < c. \tag{C.64}$$

We write $a \leq b$ as an abbreviation for "either $a < b$ or $a = b$", and we write $b > a$ to mean $a < b$. We also nest inequalities in the following way:

$$a < b \leq c = d < e$$

means

$$a < b \text{ and } b \leq c \text{ and } c = d \text{ and } d < e.$$

Addition of Inequalities: Let a, b, c, d be real numbers. Then

$$\begin{aligned}
\text{if } \ a < b \text{ and } \ c < d \quad &\text{then} \quad a + c < b + d, && \text{(C.65)}\\
\text{if } \ a \leq b \text{ and } \ c \leq d \quad &\text{then} \quad a + c \leq b + d, && \text{(C.66)}\\
\text{if } \ a < b \text{ and } \ c \leq d \quad &\text{then} \quad a + c < b + d, && \text{(C.67)}\\
\text{if } \ a < b \quad &\text{then} \quad a - c < b - c, && \text{(C.68)}\\
\text{if } \ c < d \quad &\text{then} \quad -c > -d, && \text{(C.69)}\\
\text{if } \ c < d \quad &\text{then} \quad a - c > a - d. && \text{(C.70)}
\end{aligned}$$

Multiplication of Inequalities: Let a, b, c, d be real numbers.

$$\begin{aligned}
\text{if } \ a < b \text{ and } \ c > 0 \quad &\text{then} \quad ac < bc, && \text{(C.71)}\\
\text{if } \ a < b \text{ and } \ c > 0 \quad &\text{then} \quad a/c < b/c, && \text{(C.72)}\\
\text{if } \ 0 < a < b \text{ and } \ 0 < c < d \quad &\text{then} \quad 0 < ac < bd, && \text{(C.73)}\\
\text{if } \ a < b \text{ and } \ c < 0 \quad &\text{then} \quad ac > bc, && \text{(C.74)}\\
\text{if } \ a < b \text{ and } \ c < 0 \quad &\text{then} \quad a/c > b/c, && \text{(C.75)}\\
\text{if } \ 0 < a \text{ and } \ a < b \quad &\text{then} \quad a^{-1} > b^{-1}. && \text{(C.76)}
\end{aligned}$$

Discreteness of Integers: If n is an integer, then there are no integers between n and $n + 1$, i.e. there are no integers k satisfying $n < k < n + 1$. A consequence of this is that

$$\text{If } k, n \text{ are integers, and } k < n + 1, \text{ then } k \leq n. \tag{C.77}$$

If x and y are real numbers such that $y - x > 1$ then there is an integer n such that

$$x < n < y. \tag{C.78}$$

Archimedean Property: Let x be an arbitrary real number. Then

$$\text{there exists an integer } n \text{ such that } n > x. \tag{C.79}$$

Miscellaneous Properties

Names for Rational Numbers: Every rational number r can be written as a quotient of integers:

$$r = \frac{m}{n} \text{ where } m, n \text{ are integers and } n \neq 0,$$

and without loss of generality we may take $n > 0$. In general, a rational number has many different names, e.g. $\frac{2}{3}$, $\frac{-10}{-15}$, and $\frac{34}{51}$ are different names for the same rational number. If I say "let $x = \frac{2}{3}$", I mean let x denote the rational number which has "$\frac{2}{3}$" as one of its names. You should think of each rational number as a specific point on the line of real numbers. Let m, n, p, q be integers with $n \neq 0$ and $q \neq 0$. Then

$$\frac{m}{n} = \frac{p}{q} \text{ if and only if } mq = np. \tag{C.80}$$

If n and q are *positive,* then

$$\frac{m}{n} < \frac{p}{q} \text{ if and only if } mq < np. \tag{C.81}$$

Equations C.80 and C.81 hold for arbitrary real numbers m, n, p, q. It will be assumed that if you are given two rational numbers, you can decide whether or not the first is less that the second. You also know that the sum, difference, and product of two integers is an integer, and the additive inverse of an integer is an integer.

Absolute value: If x is a real number, then the *absolute value* of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases} \tag{C.82}$$

For all real numbers x and all positive numbers a we have

$$(|x| < a) \text{ if and only if } (-a < x < a), \tag{C.83}$$
$$(|x| \leq a) \text{ if and only if } (-a \leq x \leq a). \tag{C.84}$$

For all real numbers x, y, z with $z \neq 0$,

$$|x| = |-x| \tag{C.85}$$
$$-|x| \leq x \leq |x| \tag{C.86}$$
$$|xy| = |x| \cdot |y| \tag{C.87}$$
$$\left|\frac{x}{z}\right| = \frac{|x|}{|z|}. \tag{C.88}$$

Powers: If a is a real number, and n is a non-negative integer, then the *power* a^n is defined by the rules

$$
\begin{aligned}
a^0 &= 1 && \text{(C.89)}\\
a^{n+1} &= a^n \cdot a \text{ for } n \geq 0. && \text{(C.90)}
\end{aligned}
$$

If a is a non-zero number and n is a negative integer, then a^n is defined by

$$a^n = (a^{-n})^{-1} = \frac{1}{a^{-n}}. \tag{C.91}$$

If a is a non-negative number and n is a positive integer, then $a^{\frac{1}{n}}$ is defined by

$$a^{\frac{1}{n}} \text{ is the unique non-negative number } b \text{ such that } b^n = a. \tag{C.92}$$

If a is a non-negative number and m is an arbitrary integer and n is a positive integer, then $a^{\frac{m}{n}}$ is defined by

$$a^{\frac{m}{n}} = \begin{cases} (a^{\frac{1}{n}})^m & \text{if } a > 0. \\ 0 & \text{if } a = 0 \text{ and } m > 0 \\ \text{undefined} & \text{if } a = 0 \text{ and } m < 0. \end{cases} \tag{C.93}$$

If m, n, p, q are integers such that $n \neq 0$ and $q \neq 0$ and $\frac{m}{n} = \frac{p}{q}$, then

$$(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}} = (a^p)^{\frac{1}{q}} = (a^{\frac{1}{q}})^p. \tag{C.94}$$

Monotonicity of Powers: If r is a positive rational number, and x and y are non-negative real numbers, then

$$x < y \text{ if and only if } x^r < y^r. \tag{C.95}$$

If r is a negative rational number, and x and y are positive real numbers, then

$$x < y \text{ if and only if } x^r > y^r. \tag{C.96}$$

If a is a positive real number greater than 1, and p and q are rational numbers, then

$$p < q \text{ if and only if } a^p < a^q. \tag{C.97}$$

If a is a positive real number less than 1, and p and q are rational numbers, then

$$p < q \text{ if and only if } a^p > a^q. \tag{C.98}$$

Laws of exponents: Let a and b be real numbers, and let r and s be rational numbers. Then the following relations hold whenever all of the powers involved are defined:

$$\begin{aligned} a^r a^s &= a^{r+s}, && \text{(C.99)} \\ (a^r)^s &= a^{(rs)}, && \text{(C.100)} \\ (ab)^r &= a^r b^r. && \text{(C.101)} \\ a^{-r} &= \frac{1}{a^r} && \text{(C.102)} \end{aligned}$$

Remarks on equality: If x, y and z are names for mathematical objects, then we write $x = y$ to mean that x and y are different names for the same object. Thus

$$\text{if } x = y \text{ then } y = x, \tag{C.103}$$

and it is always the case that

$$x = x. \tag{C.104}$$

It also follows that

$$\text{if } x = y \text{ and } y = z \text{ then } x = z, \tag{C.105}$$

and more generally,

$$\text{if } x = y = z = t = w \text{ then } x = w. \tag{C.106}$$

If $x = y$, then the name x can be substituted for the name y in any statement containing the name x. For example, if x, y are numbers and we know that

$$x = y, \tag{C.107}$$

then we can conclude that

$$x + 1 = y + 1, \tag{C.108}$$

and that

$$x + x = x + y. \tag{C.109}$$

When giving a proof, one ordinarily goes from an equation such as C.107 to equations such as C.108 or C.109 without mentioning the reason, and the properties C.103–C.106 are usually used without mentioning them explicitly.

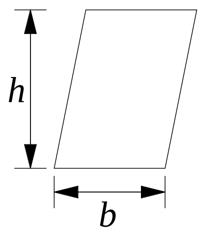
C.2 Geometrical Prerequisites

Area Formulas

It will be assumed that you are familiar with the results from Euclidean and coordinate geometry listed below.

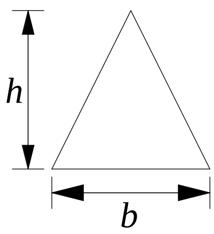
Area of a parallelogram:

$$\text{Area} = \text{base} \cdot \text{height} = bh \tag{C.110}$$



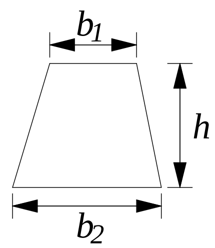
Area of a triangle:

$$\text{Area} = \tfrac{1}{2}\ \text{base} \cdot \text{height} = \tfrac{1}{2}bh \tag{C.111}$$



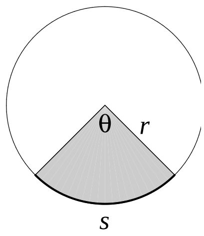
Area of a trapezoid:

$$\begin{aligned} \text{Area} &= (\text{average of bases}) \cdot \text{height} \\ &= \tfrac{1}{2}(b_1 + b_2)h \end{aligned} \tag{C.112}$$



We will always assume that angles are measured in radians unless otherwise specified. If an angle θ is inscribed in a circle of radius r and s is the length of the subtended arc, then

$$s = r\theta. \tag{C.113}$$



A right angle is $\pi/2$ and the sum of the angles of a triangle is π. When θ is four right angles in (C.113) we get

$$\text{circumference(circle)} = 2\pi r. \tag{C.114}$$

Area of a circular sector:

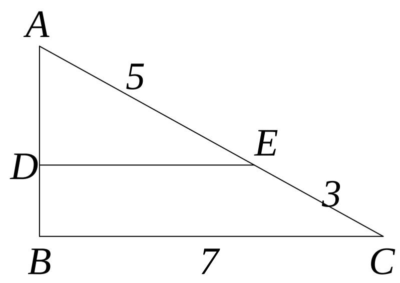
$$\begin{aligned}
\text{Area} &= \frac{1}{2} \cdot \text{radius} \cdot \text{subtended arc} \\
&= \frac{1}{2} rs \\
&= \frac{1}{2} \cdot \text{central angle} \cdot \text{radius}^2 \\
&= \frac{1}{2} \theta r^2.
\end{aligned} \tag{C.115}$$

In particular when θ is four right angles

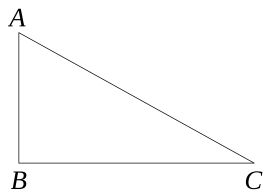
$$\text{Area(circle)} = \pi r^2. \tag{C.116}$$

Miscellaneous Properties

You should be familiar with the properties of parallel lines, and with the rules for deciding when triangles are congruent or similar. In the accompanying figure if ABC is a triangle and DE is parallel to BC, and the lengths of the sides are as labeled, you should be able to calculate DE.



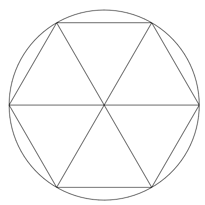
The Pythagorean Theorem: If ABC is a right triangle with the right angle at B, then



$$(AB)^2 + (BC)^2 = (AC)^2. \tag{C.117}$$

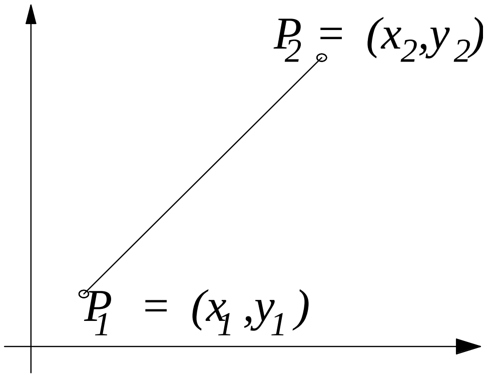
In a given circle, equal arcs subtend equal chords.

A regular hexagon inscribed in a circle has each of its sides equal to the radius of the circle. The radii joining the vertices of this hexagon to the center of the circle decompose the hexagon into six equilateral triangles.



It is assumed that you are familiar with the process of representing points in the plane by pairs of numbers. If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are points with $x_1 \neq x_2$, then the slope of the segment joining P_1 to P_2 is defined to be

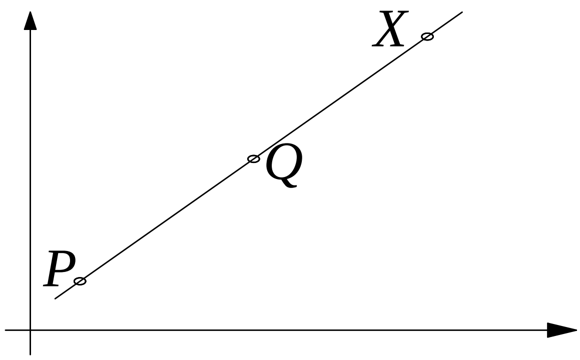
$$\text{slope}(P_1P_2) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \text{slope}(P_2P_1). \tag{C.118}$$



If $x_1 = x_2$ we say that P_1P_2 has *undefined slope*, or that P_1P_2 is a *vertical* segment. If slope(P_1P_2) is zero we say that P_1P_2 is a *horizontal* segment.

Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two distinct points in the plane. If $p_1 = q_1$, then the line passing through P and Q is defined to be the set of all points of the form (p_1, y), where y can be an arbitrary real number. If $p_1 \neq q_1$, then the *line joining P to Q* is defined to be the set consisting of P together with all points $X = (x, y)$ such that slope$(PX) =$ slope(PQ). Thus if $p_1 \neq q_1$ then (x, y) is on the line joining P to Q if and only if

$$(x, y) = (p_1, p_2) \qquad \text{or} \qquad \frac{y - p_2}{x - p_1} = \frac{q_2 - p_2}{q_1 - p_1}. \tag{C.119}$$



If $m =$ slope(PQ) then equations C.119 can be rewritten as

$$y = p_2 + m(x - p_1). \tag{C.120}$$

If $p_1 \neq q_1$ then (C.120) is called an *equation for the line joining P and Q*. If $p_1 = q_1$ then

$$x = p_1 \tag{C.121}$$

is called an equation for the line joining P and Q. Thus a point X is on the line joining P to Q if and only if the coordinates of X satisfy an equation for the line.

Two lines are *parallel*, (i.e. they do not intersect or they are identical,) if and only if they both have the same slope or they both have undefined slopes.