Chapter 6

Limits of Sequences

6.1 Absolute Value

6.1 Definition (Absolute values.) Recall that if x is a real number, then the *absolute value of* x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$

We will assume the following properties of absolute value, that follow easily from the definition:

For all real numbers x, y, z with $z \neq 0$

$$\begin{aligned} |x| &= |-x| \\ |xy| &= |x| \cdot |y| \\ \left|\frac{x}{z}\right| &= \frac{|x|}{|z|} \\ -|x| &\leq x \leq |x|. \end{aligned}$$

For all real numbers x, and all $a \in \mathbf{R}^+$

$$(|x| < a) \iff (-a < x < a)$$

and

$$(|x| \leq a) \iff (-a \leq x \leq a). \tag{6.2}$$

We also have

$$|x| \in \mathbf{R}_{\geq 0} \text{ for all } x \in \mathbf{R},$$

and

$$|x| = 0 \iff x = 0.$$

6.3 Theorem. *Let $a \in \mathbf{R}$ and let $p \in \mathbf{R}^+$. Then for all $x \in \mathbf{R}$ we have*

$$|x - a| < p \iff (a - p < x < a + p),$$

and

$$|x - a| \leq p \iff (a - p \leq x \leq a + p).$$

Equivalently, we can say that

$$\{x \in \mathbf{R} : |x - a| < p\} = (a - p, a + p)$$

and

$$\{x \in \mathbf{R} : |x - a| \leq p\} = [a - p, a + p].$$

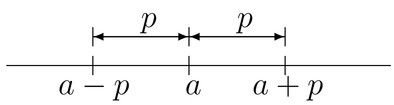
Proof: I will prove only the first statement. I have

$$\begin{aligned} |x - a| < p &\iff -p < x - a < p \\ &\iff a - p < a + (x - a) < a + p \\ &\iff a - p < x < a + p. \ \| \end{aligned}$$

6.4 Definition (Distance.) The *distance* between two real numbers x and y is defined by

$$\text{dist}(x, y) = |x - y|.$$

Theorem 6.3 says that the set of numbers whose distance from a is smaller than p is the interval $(a-p, a+p)$. Geometrically this is clear from the picture.



I remember the theorem by keeping the picture in mind.

6.5 Theorem (Triangle inequality.) *For all real numbers x and y*

$$|x + y| \leq |x| + |y|, \tag{6.6}$$

Proof For all x and y in $\mathbf{R}$ we have

$$-|x| \leq x \leq |x|$$

and

$$-|y| \leq y \leq |y|,$$

so

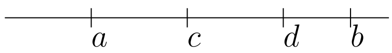
$$-(|x| + |y|) \leq x + y \leq (|x| + |y|).$$

Hence (Cf. (6.2))

$$|x + y| \leq |x| + |y|.$$

6.7 Exercise. Can you prove that for all $(x, y) \in \mathbf{R}^2 \Big(|x - y| \leq |x| - |y|\Big)$? Can you prove that for all $(x, y) \in \mathbf{R}^2 \Big(|x - y| \leq |x| + |y|\Big)$?

Remark: Let a, b, c, d be real numbers with $a < c < b$ and $a < d < b$.



Then

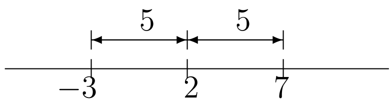
$$|c - d| < |b - a| = b - a.$$

This result should be clear from the picture. We can give an analytic proof as follows.

$$\begin{aligned}
(a < c < b \text{ and } a < d < b) &\Longrightarrow (a < c < b \text{ and } -b < -d < -a) \\
&\Longrightarrow a - b < c - d < b - a \\
&\Longrightarrow -(b - a) < c - d < (b - a) \\
&\Longrightarrow -|b - a| < c - d < |b - a| \\
&\Longrightarrow |c - d| < |b - a|.
\end{aligned}$$

6.8 Examples. Let

$$\begin{aligned}
A &= \{x \in \mathbf{R} : |x - 2| < 5\}, \\
B &= \{x \in \mathbf{R} : |x - 2| > 5\}.
\end{aligned}$$



Then a number x is in A if and only if the distance from x to 2 is smaller than 5, and x is in B if and only if the distance from x to 2 is greater than 5. I can see by inspection that

$$A = (-3, 7),$$

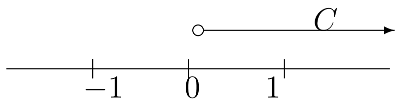
and

$$B = (-\infty, -3) \cup (7, \infty).$$

Let

$$C = \{x \in \mathbf{R} : \left|\frac{x-1}{x+1}\right| < 1\}.$$

If $x \in \mathbf{R} \setminus \{-1\}$, then x is in C if and only if $|x-1| < |x+1|$, i.e. if and only if x is closer to 1 than to -1.



I can see by inspection that the point equidistant from -1 and 1 is 0, and that the numbers that are closer to 1 than to -1 are the positive numbers, so $C = (0, \infty)$. I can also do this analytically, (but in practice I wouldn't) as follows. Since the alsolute values are all non-negative

$$\begin{aligned} |x-1| < |x+1| &\iff |x-1|^2 < |x+1|^2 \\ &\iff x^2 - 2x + 1 < x^2 + 2x + 1 \\ &\iff 0 < 4x \iff 0 < x. \end{aligned}$$

6.9 Exercise. Express each of the four sets below as an interval or a union of intervals. (You can do this problem by inspection.)

$$\begin{aligned} A_1 &= \{x \in \mathbf{R}\colon |x - \frac{1}{2}| < \frac{3}{2}\}, \\ A_2 &= \{x \in \mathbf{R}\colon |x + \frac{1}{2}| \le \frac{3}{2}\}, \\ A_3 &= \{x \in \mathbf{R}\colon |\frac{3}{2} - x| < \frac{1}{2}\}, \\ A_4 &= \{x \in \mathbf{R}\colon |\frac{3}{2} + x| \ge \frac{3}{2}\}. \end{aligned}$$

6.10 Exercise. Sketch the graphs of the functions from **R** to **R** defined by the following equations:

$$\begin{aligned} f_1(x) &= |x|, \\ f_2(x) &= |x-2|, \\ f_3(x) &= |x| - |x-2|, \\ f_4(x) &= |x| + |x-2|, \\ f_5(x) &= x^2 - 1, \\ f_6(x) &= |x^2 - 1|, \\ f_7(x) &= |x^2 - 1|^2. \end{aligned}$$

(No explanations are expected for this problem.)

6.11 Exercise. Let $f_1, \cdots, f_7$ be the functions described in the previous exercise. By looking at the graphs, express each of the following six sets in terms of intervals.

$$\begin{aligned} S_1 &= \{x \in \mathbf{R}\colon f_1(x) < 1\} \\ S_2 &= \{x \in \mathbf{R}\colon f_2(x) < 1\} \\ S_3 &= \{x \in \mathbf{R}\colon f_3(x) < 1\} \\ S_4 &= \{x \in \mathbf{R}\colon f_4(x) < 3\} \\ S_5 &= \{x \in \mathbf{R}\colon f_5(x) < 3\} \\ S_6 &= \{x \in \mathbf{R}\colon f_6(x) < 3\}. \end{aligned}$$

Let $S_7 = \{x \in \mathbf{R}\colon f_7(x) < \frac{1}{2}\}$. Represent S_7 graphically on a number line.

Remark: The notation $|x|$ for absolute value of x was introduced by Weierstrass in 1841 [15][Vol 2,page 123]. It was first introduced in connection with complex numbers. It is surprising that analysis advanced so far without introducing a special notation for this very important function.

6.2 Approximation

6.12 Definition (b approximates a.) Let ϵ be a positive number, and let a and b be arbitrary numbers. I will say that b *approximates* a *with an error smaller than* ϵ if and only if

$$|b - a| < \epsilon.$$

Remark: If b approximates a with an error smaller than ϵ, then a approximates b with an error smaller than ϵ, since $|a-b| = |b-a|$.

6.13 Definition (Approximation to n decimals.) Let $n \in \mathbf{Z}^+$, and let a, b be real numbers. I will say that b *approximates a with n decimal accuracy* if and only if b approximates a with an error smaller than $\frac{1}{2} \cdot 10^{-n}$; i.e., if and only if

$$|b-a| < \frac{1}{2} 10^{-n}.$$

6.14 Notation. If I write three dots $(\cdots)$ at the end of a number written in decimal notation, I assume that all of the digits before the three dots are correct. Thus since $\pi = 3.141592653589\cdots$, I have $\pi = 3.1415\cdots$, and $\pi = 3.1416$ with 4 decimal accuracy.

6.15 Example.

$$\pi = 3.141592653589793\cdots$$

and

$$\frac{22}{7} = 3.142857142857142\cdots.$$

Hence

$$3.1415 < \pi < \frac{22}{7} < 3.1429,$$

and

$$\left|\frac{22}{7} - \pi\right| < 3.1429 - 3.1415 = .0014 < .005 = \frac{1}{2} \cdot 10^{-2}.$$

Hence $\frac{22}{7}$ approximates π with an error smaller than .0014, and $\frac{22}{7}$ approximates π with 2 decimal accuracy.

6.16 Example. We see that

.49 approximates .494999 with 2 decimal accuracy,

and

.50 approximates .495001 with 2 decimal accuracy,

but there is no two digit decimal that approximates .495000 with 2 decimal accuracy.

6.17 Example. Since

$$|.49996 - .5| = .00004 < .00005 = \frac{1}{2} \cdot 10^{-4},$$

we see that .5 approximates .49996 with 4 decimal accuracy, even though the two numbers have no decimal digits in common. Since

$$|.49996 - .4999| = .00006 > \frac{1}{2} \cdot 10^{-4},$$

we see that .4999 does not approximate .49996 with 4 decimal accuracy, even though the two numbers have four decimal digits in common.

6.18 Theorem (Strong approximation theorem.) *Let a and b be real numbers. Suppose that for every positive number ϵ, b approximates a with an error smaller than ϵ. Then $b = a$.*

Proof: Suppose that b approximates a with an error smaller than ϵ for every positive number ϵ. Then

$$|b - a| < \epsilon \text{ for every } \epsilon \text{ in } \mathbf{R}^+.$$

Hence

$$|b - a| \neq \epsilon \text{ for every } \epsilon \text{ in } \mathbf{R}^+,$$

i.e., $|b - a| \notin \mathbf{R}^+$. But $|b - a| \in \mathbf{R}_{\geq 0}$, so it follows that $|b - a| = 0$, and consequently $b - a = 0$; i.e., $b = a$. |||

6.3 Convergence of Sequences

6.19 Definition ($\{a_n\}$ converges to L.)

Let $\{a_n\}$ be a sequence of real numbers, and let L be a real number. We say that $\{a_n\}$ *converges to* L if for every positive number ϵ there is a number $N(\epsilon)$ in $\mathbf{Z}^+$ such that all of the numbers a_n for which $n \geq N(\epsilon)$ approximate L with an error smaller than ϵ. We denote the fact that $\{a_n\}$ converges to L by the notation

$$\{a_n\} \to L.$$

Thus "$\{a_n\} \to L$" means:

For every $\epsilon \in \mathbf{R}^+$ there is a number $N(\epsilon)$ in $\mathbf{Z}^+$ such that

$$|a_n - L| < \epsilon \text{ for all } n \text{ in } \mathbf{Z}^+ \text{ with } n \geq N(\epsilon).$$

Since

$$|a_n - L| = |(a_n - L) - 0| = \Big||a_n - L| - 0\Big|,$$

it follows immediately from the definition of convergence that

$$(\{a_n\} \to L) \iff (\{a_n - L\} \to 0) \iff (|a_n - L| \to 0).$$

We will make frequent use of these equivalences.

6.20 Example. If $a \in \mathbf{R}^+$ then

$$\left\{a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\right\} \to a^3.$$

Proof: Let ϵ be a generic element of $\mathbf{R}^+$. I must find a number $N(\epsilon)$ such that

$$\left|a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) - a^3\right| < \epsilon \tag{6.21}$$

whenever $n \geq N(\epsilon)$. Well, for all n in $\mathbf{Z}^+$

$$\begin{aligned} \left|a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) - a^3\right| &= \left|a^3\left(1+\frac{3}{2n}+\frac{1}{2n^2}\right) - a^3\right| \\ &= \left|a^3\left(\frac{3}{2n}+\frac{1}{2n^2}\right)\right| = a^3\left(\frac{3}{2n}+\frac{1}{2n^2}\right) \\ &\leq a^3\left(\frac{3}{2n}+\frac{1}{2n}\right) = \frac{2a^3}{n}. \end{aligned} \tag{6.22}$$

Now for every n in $\mathbf{Z}^+$ we have

$$\left(\frac{2a^3}{n} < \epsilon\right) \iff \left(\frac{2a^3}{\epsilon} < n\right),$$

and by the Archimedean property of $\mathbf{R}$ there is some integer $N(\epsilon)$ such that $\dfrac{2a^3}{\epsilon} < N(\epsilon)$. For all $n \geq N(\epsilon)$ we have

$$(n \geq N(\epsilon)) \implies \left(\frac{2a^3}{\epsilon} < N(\epsilon) \leq n\right) \implies \left(\left(\frac{2a^3}{n}\right) < \epsilon\right),$$

so by (6.22)

$$(n \geq N(\epsilon)) \implies \left|a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)-a^3\right| \leq \frac{2a^3}{n} < \epsilon.$$

Hence by the definition of convergence we have

$$\left\{a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\right\} \to a^3. \ \| \tag{6.23}$$

A very similar argument can be used to show that

$$\left\{a^3\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\right\} \to a^3. \tag{6.24}$$

6.25 Example. In the eighteenth century the rather complicated argument just given would have been stated as

> If n is infinitely large, then $a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) = a^3$.

The first calculus text book (written by Guillaume François de l'Hôpital and published in 1696) sets forth the postulate

> Grant that two quantities, whose difference is an infinitely small quantity, may be taken (or used) indifferently for each other: or (which is the same thing) that a quantity which is increased or decreased only by an infinitely small quantity, may be considered as remaining the same[35, page 314].

If n is infinite, then $\frac{1}{n}$ is infinitely small, so $\left(1+\frac{1}{n}\right) = 1$, and similarly $\left(1+\frac{1}{2n}\right) = 1$. Hence

$$a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) = a^3 \cdot 1 \cdot 1 = a^3.$$

There were numerous objections to this sort of reasoning. Even though $\left(1+\frac{1}{n}\right) = 1$, we do not have $\left(1+\frac{1}{n}\right) - 1 = 0$, since

$$\frac{\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}} = 1.$$

It took many mathematicians working over hundreds of years to come up with our definition of convergence.

6.26 Theorem (Uniqueness theorem for convergence.) *Let $\{a_n\}$ be a sequence of real numbers, and let a, b be real numbers. Suppose*

$$\{a_n\} \to a \text{ and } \{a_n\} \to b.$$

Then $a = b$.

Proof: Suppose $\{a_n\} \to a$ and $\{a_n\} \to b$. By the triangle inequality

$$|a - b| = |(a - a_n) - (b - a_n)| \leq |a - a_n| + |b - a_n|. \tag{6.27}$$

Let ϵ be a generic element of $\mathbf{R}^+$. Then $\frac{\epsilon}{2}$ is also in $\mathbf{R}^+$. Since $\{a_n\} \to a$, there is a number $N(\frac{\epsilon}{2})$ in $\mathbf{Z}^+$ such that

$$|a - a_n| < \frac{\epsilon}{2} \text{ for all } n \geq N(\tfrac{\epsilon}{2}). \tag{6.28}$$

Since $\{a_n\} \to b$ there is a number $M(\frac{\epsilon}{2})$ in $\mathbf{Z}^+$ such that

$$|b - a_n| < \frac{\epsilon}{2} \text{ for all } n \geq M(\tfrac{\epsilon}{2}). \tag{6.29}$$

Let $P(\epsilon)$ be the larger of $N(\frac{\epsilon}{2})$ and $M(\frac{\epsilon}{2})$. If n is a positive integer and $n \geq P(\epsilon)$ then by (6.27), (6.28), and (6.29), we have

$$|a - b| \leq |a - a_n| + |b - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since this holds for all ϵ in $\mathbf{R}^+$, we have $a = b$. |||

6.30 Definition (Limit of a sequence.) Let $\{a_n\}$ be a sequence of real numbers. If there is a number a such that $\{a_n\} \to a$, we write $\lim\{a_n\} = a$. The uniqueness theorem for convergence shows that this definition makes sense. If $\lim\{a_n\} = a$, we say *a is the limit of the sequence $\{a_n\}$*.

6.31 Definition (Convergent and divergent sequence.) Let $\{a_n\}$ be a sequence of real numbers. If there is a number a such that $\{a_n\} \to a$, we say that $\{a_n\}$ is a *convergent sequence*. If there is no such number a, we say that $\{a_n\}$ is a *divergent sequence*.

6.32 Example. It follows from example 6.20 that

$$\lim\left\{a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\right\}=a^3$$

for all a in $\mathbf{R}^+$. Hence $\left\{a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\right\}$ is a convergent sequence for each a in $\mathbf{R}^+$.

The sequence $\{n\}$ is a divergent sequence. To see this, suppose there were a number a such that $\{n\}\to a$.

Then we can find a number $N(\frac{1}{3})$ such that

$$|n-a|<\frac{1}{3} \text{ for all } n\geq N(\tfrac{1}{3}).$$

In particular

$$\left|N(\tfrac{1}{3})-a\right|<\frac{1}{3} \text{ and } \left|\left(N(\tfrac{1}{3})+1\right)-a\right|<\frac{1}{3}$$

(since $N(\frac{1}{3})+1$ is an integer greater than $N(\frac{1}{3})$). Hence, by the triangle inequality

$$\begin{aligned} 1 &= |1|=\left|\left(N(\tfrac{1}{3})+1-a\right)-\left(N(\tfrac{1}{3})-a\right)\right| \\ &\leq \left|N(\tfrac{1}{3})+1-a\right|+\left|N(\tfrac{1}{3})-a\right|<\frac{1}{3}+\frac{1}{3}=\frac{2}{3} \end{aligned}$$

i.e., $1<\frac{2}{3}$ which is false.

Since the assumption $\{n\}\to a$ has led to a contradiction, it is false that $\{n\}\to a$. ‖

6.33 Exercise. Let $\{a_n\}$ be a sequence of real numbers, and let a be a real number. Suppose that as n gets larger and larger, a_n gets nearer and nearer to a, i.e., suppose that for all m and n in $\mathbf{Z}^+$

$$(n>m) \implies (|a_n-a|<|a_m-a|).$$

Does it follow that $\{a_n\}$ converges to a?

6.34 Exercise. For each of the sequences below, calculate the first few terms, and make a guess as to whether or not the sequence converges. In some cases you will need to use a calculator. Try to explain the basis for your guess. (If you can prove your guess is correct, do so, but in several cases the proofs involve more mathematical knowledge than you now have.)

$\{a_n\} = \{(-1)^n\}.$

$$\{c_n\} = \left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n}\right\}.$$

$$\{d_n\} = \left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right\}$$

This problem was solved by Leonard Euler (1707-1783)[18, pp138-139].

$$\{e_n\} = \left\{1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}\right\}.$$

$$\{f_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}$$

This problem was solved by Jacob Bernoulli (1654-1705)[8, pp94-97].

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6.4 Properties of Limits.

In this section I will state some basic properties of limits. All of the statements listed here as assumptions are, in fact, theorems that can be proved from the definition of limits. I am omitting the proofs because of lack of time, and because the results are so plausible that you will probably believe them without a proof.

6.35 Definition (Constant sequence.) If r is a real number then the sequence $\{r\}$ all of whose terms are equal to r is called a *constant sequence*

$$\{r\} = \{r, r, r, \cdots\}.$$

It is an immediate consequence of the definition of convergence that

$$\{r\} \to r$$

for every real number r. (If $r_n = r$ for all n in $\mathbf{Z}^+$ then $|r_n - r| = 0 < \epsilon$ for all ϵ in $\mathbf{R}^+$ so r_n approximates r with an error smaller than ϵ for all $n \geq 1$. $\|$.)

We have just proved

6.36 Theorem (Constant sequence rule.) *If $\{r\}$ denotes a constant sequence of real numbers, then*

$$\lim\{r\} = r.$$

6.37 Theorem (Null sequence rule.) *Let α be a positive rational number. Then*

$$\lim\left\{\frac{1}{n^\alpha}\right\} = 0.$$

Proof: Let α be a positive rational number, and let ϵ be a generic positive number. By the monotonicity of powers (see (C.95) in appendix C), we have

$$\begin{aligned}\frac{1}{n^\alpha} < \epsilon \quad &\iff \quad \left(\frac{1}{n^\alpha}\right)^{\frac{1}{\alpha}} < (\epsilon)^{\frac{1}{\alpha}} \iff \frac{1}{n} < \epsilon^{\left(\frac{1}{\alpha}\right)} \\ &\iff \quad n > \frac{1}{\epsilon^{\frac{1}{\alpha}}} = \epsilon^{-\frac{1}{\alpha}}.\end{aligned}$$

By the Archimedean property for $\mathbf{R}$ there is an integer $N(\epsilon)$ in $\mathbf{Z}^+$ such that

$$N(\epsilon) > \epsilon^{-\frac{1}{\alpha}}.$$

Then for all n in $\mathbf{Z}^+$

$$n \geq N(\epsilon) \implies n \geq \epsilon^{-1/\alpha} \implies \frac{1}{n^\alpha} < \epsilon \implies \left|\frac{1}{n^\alpha} - 0\right| < \epsilon.$$

Thus $\lim\left\{\frac{1}{n^\alpha}\right\} = 0$. |||

6.38 Assumption (Sum rule for sequences.) *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences of real numbers. Then*

$$\lim\{a_n + b_n\} = \lim\{a_n\} + \lim\{b_n\}$$

and

$$\lim\{a_n - b_n\} = \lim\{a_n\} - \lim\{b_n\}.$$

The sum rule is actually easy to prove, but I will not prove it. (You can probably supply a proof for it.)

Notice the hypothesis that $\{a_n\}$ and $\{b_n\}$ are *convergent* sequences. It is not true in general that

$$\lim\{a_n + b_n\} = \lim\{a_n\} + \lim\{b_n\}.$$

For example, the statement

$$\lim\{(-1)^n + (-1)^{n+1}\} = \lim\{(-1)^n\} + \lim\{(-1)^{n+1}\}$$

is false, since

$$\lim\{(-1)^n + (-1)^{n+1}\} = \lim\{0\} = 0$$

but neither of the limits $\lim\{(-1)^n\}$ or $\lim\{(-1)^{n+1}\}$ exist.

6.39 Assumption (Product rule for sequences.) *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Then*

$$\lim\{a_n \cdot b_n\} = \lim\{a_n\} \cdot \lim\{b_n\}.$$

An important special case of the product rule occurs when one of the sequences is constant: If a is a real number, and $\{b_n\}$ is a convergent sequence, then

$$\lim\{ab_n\} = a \lim\{b_n\}.$$

The intuitive content of the product rule is that if a_n approximates a very well, and b_n approximates b very well, then $a_n b_n$ approximates ab very well. It is somewhat tricky to prove this for a reason that is illustrated by the following example.

According to Maple,

$$\sqrt{99999999} = 9999.99994999999987499\cdots$$

so 9999.9999 approximates $\sqrt{99999999}$ with 4 decimal accuracy. Let

$$a = b = 9999.9999,$$

and let

$$A = B = \sqrt{99999999}.$$

Then a approximates A with 4 decimal accuracy and b approximates B with 4 decimal accuracy. But

$$AB = 99999999$$

and

$$ab = 99999998.00000001$$

so ab does not approximate AB with an accuracy of even one decimal.

6.40 Assumption (Quotient rule for sequences.) *Let $\{a_n\}$ and $\{b_n\}$ be convergent real sequences such that $b_n \neq 0$ for all n in $\mathbf{Z}^+$ and $\lim\{b_n\} \neq 0$. Then*

$$\lim\left\{\frac{a_n}{b_n}\right\} = \frac{\lim\{a_n\}}{\lim\{b_n\}}.$$

The hypotheses here are to be expected. If some term b_n were zero, then $\left\{\frac{a_n}{b_n}\right\}$ would not be a sequence, and if $\lim\{b_n\}$ were zero, then $\frac{\lim\{a_n\}}{\lim\{b_n\}}$ would not be defined.

6.41 Assumption (Inequality rule for sequences.) *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Suppose there is an integer N in $\mathbf{Z}^+$ such that*

$$a_n \leq b_n \text{ for all } n \text{ in } \mathbf{Z}_{\geq N}.$$

Then

$$\lim\{a_n\} \leq \lim\{b_n\}.$$

The most common use of this rule is in situations where

$$0 \leq b_n \text{ for all } n$$

and we conclude that

$$0 \leq \lim\{b_n\}.$$

6.42 Assumption (Squeezing rule for sequences.) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three real sequences. Suppose there is an integer N in $\mathbf{Z}^+$ such that*

$$a_n \leq b_n \leq c_n \text{ for all } n \in \mathbf{Z}_{\geq N}. \tag{6.43}$$

Suppose further, that $\{a_n\}$ and $\{c_n\}$ both converge to the same limit L. Then $\{b_n\}$ also converges to L.

If we knew that the middle sequence, $\{b_n\}$ in the squeezing rule was convergent, then we would be able to prove the squeezing rule from the inequality rule, since if all three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ converge, then it follows from (6.43) that

$$\lim\{a_n\} \leq \lim\{b_n\} \leq \lim\{c_n\},$$

i.e.

$$L \leq \lim\{b_n\} \leq L$$

and hence $\lim\{b_n\} = L$. The power of the squeezing rule is that it allows us to conclude that a limit exists.

6.44 Definition (Translate of a sequence.) Let $\{a_n\}$ be a real sequence, and let $p \in \mathbf{Z}^+$. The sequence $\{a_{n+p}\}$ is called a *translate* of $\{a_n\}$.

6.45 Example. If

$$\{a_n\} = \left\{\frac{1}{n^2}\right\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \cdots\right\}$$

then

$$\{a_{n+2}\} = \left\{\frac{1}{(n+2)^2}\right\} = \left\{\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \cdots\right\}.$$

If

$$\{b_n\} = \{(-1)^n\}$$

then

$$\{b_{n+2}\} = \{(-1)^{n+2}\} = \{(-1)^n\} = \{b_n\}.$$

6.46 Theorem (Translation rule for sequences.) *Let $\{a_n\}$ be a convergent sequence of real numbers, and let p be a positive integer. Then $\{a_{n+p}\}$ is convergent and*

$$\lim\{a_{n+p}\} = \lim\{a_n\}.$$

Proof: Suppose $\lim\{a_n\} = a$, and let ϵ be a generic element in $\mathbf{R}^+$. Then we can find an integer $N(\epsilon)$ in $\mathbf{Z}^+$ such that

$$|a_n - a| < \epsilon \text{ for all } n \text{ in } \mathbf{Z}^+ \text{ with } n \geq N(\epsilon).$$

If $n \geq N(\epsilon)$ then $n + p \geq N(\epsilon) + p \geq N(\epsilon)$ so

$$|a_{n+p} - a| < \epsilon.$$

This shows that $\lim\{a_{n+p}\} = a = \lim\{a_n\}$. ‖

6.47 Example. The sequence

$$\{a_n\} = \left\{\frac{1}{n+4}\right\} = \left\{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \cdots\right\}$$

is a translate of the sequence $\left\{\frac{1}{n}\right\}$. Since $\lim\left\{\frac{1}{n}\right\} = 0$ it follows from the translation theorem that $\lim\left\{\frac{1}{n+4}\right\} = 0$ also.

6.48 Theorem (nth root rule for sequences.) *Let a be a positive number then*

$$\lim\left\{a^{\frac{1}{n}}\right\} = 1.$$

Proof: Case 1: Suppose $a = 1$. Then

$$\lim\left\{a^{\frac{1}{n}}\right\} = \lim\{1\} = 1.$$

Case 2: Suppose $a > 1$, so that $a^{\frac{1}{n}} > 1$ for all $n \in \mathbf{Z}^+$. Let ϵ be a generic positive number, and let n be a generic element of $\mathbf{Z}^+$. Since ln is strictly increasing on $\mathbf{R}^+$ we have

$$\begin{aligned} \left(a^{\frac{1}{n}} - 1 < \epsilon\right) &\iff \left(a^{\frac{1}{n}} < 1 + \epsilon\right) \iff \left(\ln(a^{\frac{1}{n}}) < \ln(1+\epsilon)\right) \\ &\iff \frac{1}{n}\ln(a) < \ln(1+\epsilon) \\ &\iff \frac{\ln(a)}{\ln(1+\epsilon)} < n. \end{aligned} \tag{6.49}$$

(In the last step I used the fact that $\ln(1+\epsilon) > 0$ if $\epsilon > 0$.) By the Archimedean property for $\mathbf{R}$ there is an integer $N(\epsilon)$ in $\mathbf{Z}^+$ such that

$$\frac{\ln(a)}{\ln(1+\epsilon)} < N(\epsilon).$$

For all $n \in \mathbf{Z}^+$ we have

$$\begin{aligned} n \geq N(\epsilon) &\implies \frac{\ln(a)}{\ln(1+\epsilon)} < N(\epsilon) \leq n \\ &\implies a^{\frac{1}{n}} - 1 < \epsilon \implies \left|a^{\frac{1}{n}} - 1\right| < \epsilon. \end{aligned}$$

Hence $\lim \left\{a^{\frac{1}{n}}\right\} = 1$.
Case 3: Suppose $0 < a < 1$. Then $a^{-1} > 1$ so by Case 2, we have

$$\begin{aligned}\lim \left\{a^{\frac{1}{n}}\right\} &= \lim \left\{\frac{1}{(a^{-1})^{\frac{1}{n}}}\right\} \\ &= \frac{\lim\{1\}}{\lim \left\{(a^{-1})^{\frac{1}{n}}\right\}} = \frac{1}{1} = 1.\end{aligned}$$

Thus, in all cases, we have

$$\lim \left\{a^{\frac{1}{n}}\right\} = 1. \parallel$$

6.5 Illustrations of the Basic Limit Properties.

6.50 Example. In example 6.20, we used the definition of limit to show that

$$\lim \left\{a^3 \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right)\right\} = a^3$$

for all $a \in \mathbf{R}^+$, and claimed that a similar argument shows that

$$\lim \left\{a^3 \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)\right\} = a^3 \tag{6.51}$$

We will now use the basic properties of limits to prove (6.51). By the product rule and the null sequence rule,

$$\lim \left\{\frac{1}{2n}\right\} = \lim \left\{\frac{1}{2} \cdot \frac{1}{n}\right\} = \frac{1}{2} \lim \left\{\frac{1}{n}\right\} = \frac{1}{2} \cdot 0 = 0.$$

Hence by the sum rule

$$\lim \left\{1 - \frac{1}{2n}\right\} = \lim\{1\} - \lim \left\{\frac{1}{2n}\right\} = 1 - 0 = 1.$$

By the sum rule and the null sequence rule

$$\lim \left\{1 - \frac{1}{n}\right\} = \lim\{1\} - \lim \left\{\frac{1}{n}\right\} = 1 - 0 = 1.$$

Hence by the product rule,

$$\begin{aligned}\lim \left\{\left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{1}{2n}\right)\right\} &= \lim \left\{\left(1 - \frac{1}{n}\right)\right\} \cdot \lim \left\{\left(1 - \frac{1}{2n}\right)\right\} \\ &= 1 \cdot 1 = 1.\end{aligned}$$

Now $\{a^3\}$ is a constant sequence, so by the product rule,

$$\begin{aligned}\lim\left\{a^3\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\right\} &= a^3\cdot\lim\left\{\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{1}{2n}\right)\right\}\\ &= a^3\cdot 1 = a^3.\end{aligned}$$

6.52 Example. In the previous example, I made at least eight applications of our limit rules. However, the applications are completely mechanical so I will usually not be so careful, and in a situation like this, I will just write

$$\lim\left\{a^3\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\right\} = a^3\cdot(1-0)\cdot\left(1-\frac{1}{2}\cdot 0\right) = a^3. \tag{6.53}$$

The argument given in equation (6.53) looks remarkably similar to the eighteenth century argument given in example 6.25.

6.54 Example. Let a be a positive number, and let

$$A(a) = \text{area}(\{(x,y)\in\mathbf{R}^2 \colon 0\le x\le a \text{ and } 0\le y\le x^2\}).$$

In (2.13), we showed that

$$\frac{a^3}{3}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\le\ A(a)\le\frac{a^3}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right), \tag{6.55}$$

for all $n\in\mathbf{Z}^+$, and claimed that these inequalities show that $A(a)=\dfrac{a^3}{3}$. Now I want to examine the claim more closely.

In example 6.20 we proved that

$$\lim\left\{a^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\right\} = a^3,$$

and in example 6.50 we proved that

$$\lim\left\{a^3\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\right\} = a^3$$

By applying the squeezing rule to equation 6.55, we see that

$$\lim\{A(a)\} = a^3,$$

i.e.

$$A(a) = \frac{a^3}{3}.\ \|$$

6.56 Example. I will now consider the limit

$$\lim\left\{\frac{n^2-2n}{n^2+3n}\right\}.$$

Here I cannot apply the quotient rule for sequences, because the limits of the numerator and denominator do not exist. However, I notice that I can simplify my sequence:

$$\left\{\frac{n^2-2n}{n^2+3n}\right\}=\left\{\frac{n-2}{n+3}\right\}.$$

I will now use a trick. I will factor the highest power of n out of the numerator and denominator:

$$\left\{\frac{n-2}{n-3}\right\}=\left\{\frac{n\left(1-\frac{2}{n}\right)}{n\left(1-\frac{3}{n}\right)}\right\}=\left\{\frac{1-\frac{2}{n}}{1-\frac{3}{n}}\right\}.$$

It is now clear what the limit is.

$$\lim\left\{\frac{n^2-2n}{n^2+3n}\right\}=\lim\left\{\frac{1-\frac{2}{n}}{1-\frac{3}{n}}\right\}=\frac{1-2\cdot 0}{1-3\cdot 0}=1.$$

6.57 Example. I want to investigate

$$\lim\left\{\frac{n-2n^2+3}{4+6n+n^2}.\right\}$$

I'll apply the factoring trick of the previous example.

$$\left\{\frac{n-2n^2+3}{4+6n+n^2}\right\}=\left\{\frac{n^2\left(\frac{1}{n}-2+\frac{3}{n^2}\right)}{n^2\left(\frac{4}{n^2}+\frac{6}{n}+1\right)}\right\}=\left\{\frac{\frac{1}{n}-2+\frac{3}{n^2}}{\frac{4}{n^2}+\frac{6}{n}+1}\right\}$$

so

$$\begin{aligned}\lim\left\{\frac{n-2n^2+3}{4+6n+n^2}\right\} &= \lim\left\{\frac{\frac{1}{n}-2+\frac{3}{n^2}}{\frac{4}{n^2}+\frac{6}{n}+1}\right\}=\frac{0-2+3\cdot 0}{4\cdot 0+6\cdot 0+1}\\ &= -2.\end{aligned}$$

6.58 Example. I want to find

$$\lim\left\{\frac{1}{n+4}\right\}.$$

I observe that $\left\{\dfrac{1}{n+4}\right\}$ is a translate of $\left\{\dfrac{1}{n}\right\}$ so by the translation rule

$$\lim\left\{\frac{1}{n+4}\right\} = \lim\left\{\frac{1}{n}\right\} = 0.$$

I can also try to do this by my factoring trick:

$$\begin{aligned}\lim\left\{\frac{1}{n+4}\right\} &= \lim\left\{\frac{1}{n\left(1+\frac{4}{n}\right)}\right\} = \lim\left\{\frac{\frac{1}{n}}{1+\frac{4}{n}}\right\} \\ &= \frac{0}{1+4\cdot 0} = 0.\end{aligned}$$

6.59 Exercise. Find the following limits, or explain why they don't exist.

a) $\lim\left\{7+\dfrac{6}{n}+\dfrac{8}{\sqrt{n}}\right\}$

b) $\lim\left\{\dfrac{4+\frac{1}{n}}{5+\frac{1}{n}}\right\}$

c) $\lim\left\{\dfrac{3n^2+n+1}{1+3n+4n^2}\right\}$

d) $\lim\left\{\dfrac{\left(2+\frac{1}{n}\right)^2+4}{\left(2+\frac{1}{n}\right)^3+8}\right\}$

e) $\lim\left\{\dfrac{\left(2+\frac{1}{n}\right)^2-4}{\left(2+\frac{1}{n}\right)^3-8}\right\}$

f) $\lim\left\{\dfrac{8n^3+13n}{17+12n^3}\right\}$

g) $\lim\left\{\dfrac{8(n+4)^3+13(n+4)}{17+12(n+4)^3}\right\}$

h) $\lim\left\{\dfrac{n+1}{n^2+1}\right\}$.

6.60 Example. Let a be a real number greater than 1, and let

$$S_a = \{(x,y) \in \mathbf{R}^2 \colon 1 \le x \le a \text{ and } 0 \le y \le \frac{1}{x^2}\}.$$

In (2.34) we showed that

$$\frac{(1-a^{-1})}{a^{\frac{1}{n}}} \le \text{area}(S_a) \le a^{\frac{1}{n}}(1-a^{-1}) \text{ for all } n \in \mathbf{Z}^+. \tag{6.61}$$

I want to conclude from this that $\text{area}(S_a) = (1-a^{-1})$.

By the nth root rule, and the quotient and product rules, we have

$$\lim\left\{\frac{(1-a^{-1})}{a^{\frac{1}{n}}}\right\} = \frac{\lim\{1-a^{-1}\}}{\lim\{a^{\frac{1}{n}}\}} = \frac{(1-a^{-1})}{1} = (1-a^{-1}),$$

and

$$\lim\left\{a^{\frac{1}{n}}(1-a^{-1})\right\} = \lim\left\{a^{\frac{1}{n}}\right\}\lim\{(1-a^{-1})\} = 1\cdot(1-a^{-1}) = (1-a^{-1}).$$

By (6.61) and the squeezing rule, we conclude that

$$\lim\{\text{area}(S_a)\} = (1-a^{-1}),$$

i.e.

$$\text{area}(S_a) = (1-a^{-1}).$$

6.62 Example. Let the sequence $\{a_n\}$ be defined by the rules

$$\begin{aligned} a_1 &= 1, \\ a_{n+1} &= \frac{a_n^2+2}{2a_n} \text{ for } n \ge 1. \end{aligned} \tag{6.63}$$

Thus, for example

$$a_2 = \frac{1+2}{2} = \frac{3}{2}$$

and

$$a_3 = \frac{\frac{9}{4}+2}{3} = \frac{17}{12}.$$

It is clear that $a_n > 0$ for all n in $\mathbf{Z}^+$. Let $L = \lim\{a_n\}$. Then by the translation rule, $L = \lim\{a_{n+1}\}$ also. From (6.63) we have

$$2a_n a_{n+1} = a_n^2 + 2 \text{ for all } n \in \mathbf{Z}_{\ge 2}.$$

Thus

$$\lim\{2a_n a_{n+1}\} = \lim\{a_n^2 + 2\},$$

i.e.

$$2 \cdot \lim\{a_n\} \lim\{a_{n+1}\} = \lim\{a_n\}^2 + \lim\{2\}.$$

Hence

$$2 \cdot L \cdot L = L^2 + 2.$$

Thus $L^2 = 2$, and it follows that $L = \sqrt{2}$ or $L = -\sqrt{2}$. But we noticed above that $a_n > 0$ for all n in $\mathbf{Z}^+$, and hence by the inequality rule for sequences, $L \geq 0$. Hence we conclude that $L = \sqrt{2}$, i.e.,

$$\lim\{a_n\} = \sqrt{2}. \tag{6.64}$$

(Actually there is an error in the reasoning here, which you should try to find, but the conclusion (6.64) is in fact correct. After you have done exercise 6.68, the error should become apparent.)

6.65 Exercise. Use a calculator to find the first six terms of the sequence (6.63). Do all calculations using all the accuracy your calculator allows, and write down the results to all the accuracy you can get. Compare your answers with $\sqrt{2}$ (as given by your calculator) and for each term note how many decimal places accuracy you have.

6.66 Example. Let $\{b_n\}$ be the sequence defined by the rules

$$\begin{aligned} b_1 &= 1, \\ b_2 &= 1, \\ b_n &= \frac{1 + b_{n-1}}{b_{n-2}} \quad \text{for} \quad n > 2. \end{aligned} \tag{6.67}$$

Thus, for example

$$b_3 = \frac{1+1}{1} = 2$$

and

$$b_4 = \frac{1+2}{1} = 3.$$

Notice that $b_n > 0$ for all n. Let

$$L = \lim\{b_n\}.$$

By the translation rule

$$L = \lim\{b_{n+1}\} \quad \text{and} \quad L = \lim\{b_{n+2}\}.$$

By (6.67) (with n replaced by $n+2$), we have

$$b_n b_{n+2} = 1 + b_{n+1} \quad \text{for all} \quad n \quad \text{in} \quad \mathbf{Z}^+.$$

Hence

$$\begin{aligned} L^2 &= \lim\{b_n\} \cdot \lim\{b_{n+2}\} \\ &= \lim\{b_n b_{n+2}\} \\ &= \lim\{1 + b_{n+1}\} = 1 + L. \end{aligned}$$

Thus

$$L^2 - L - 1 = 0.$$

By the quadratic formula

$$L = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad L = \frac{1-\sqrt{5}}{2}.$$

Since $b_n > 0$ for all n, we have $L \geq 0$, so we have

$$L = \frac{1+\sqrt{5}}{2}.$$

(This example has the same error as the previous one.)

6.68 Exercise. Repeat exercise 6.65 using the sequence $\{b_n\}$ described in (6.67) in place of the sequence $\{a_n\}$, and $\left(\frac{1+\sqrt{5}}{2}\right)$ in place of $\sqrt{2}$. After doing this problem, you should be able to point out the error in examples (6.62) and (6.66). (This example is rather surprising. I took it from [14, page 55, exercise 20].)

6.69 Exercise. For each of the statements below: if the statement is false, give a counterexample; if the statement is true, then justify it by means of limit rules we have discussed.

a) Let $\{a_n\}$ be a convergent sequence of real numbers. If $a_n > 0$ for all n in $\mathbf{Z}^+$, then $\lim\{a_n\} > 0$.

b) Let $\{a_n\}$ and $\{b_n\}$ be real sequences. If $\lim\{a_n\} = 0$, then $\lim\{a_n b_n\} = 0$.

c) Let $\{a_n\}$ be a real sequence. If $\lim\{a_n^2\} = 1$ then either $\lim\{a_n\} = 1$ or $\lim\{a_n\} = -1$.

d) Let $\{a_n\}$ and $\{b_n\}$ be real sequences. If $\lim\{a_n b_n\} = 0$, then either $\lim\{a_n\} = 0$ or $\lim\{b_n\} = 0$.

6.70 Exercise. Let a and r be positive numbers and let

$$S_0^a[t^r] = \{(x, y) \in \mathbf{R}^2 \colon 0 \leq x \leq a \ \text{ and } \ 0 \leq y \leq x^r\}.$$

In (2.4) we showed that

$$\frac{a^{r+1}}{n^{r+1}}\,(1^r + 2^r + \cdots + (n-1)^r) \leq \alpha(S_0^a[t^r]) \leq \frac{a^{r+1}}{n^{r+1}}(1^r + 2^r + \cdots + n^r).$$

Use this result, together with Bernoulli's power sums listed on page 27 to find the area of $S_0^a[t^3]$.

6.71 Theorem (nth power theorem.) *Let r be a real number such that $|r| < 1$. Then $\lim\{r^n\} = 0$.*

Proof: Let $L = \lim\{r^{n-1}\}$. Now $\{r^n\}$ is a translate of $\{r^{n-1}\}$, so by the translation theorem

$$\begin{aligned} L &= \lim\{r^{n-1}\} = \lim\{r^n\} = \lim\{r \cdot r^{n-1}\} \\ &= \lim\{r\} \lim\{r^{n-1}\} = rL \end{aligned}$$

so we have $L - rL = 0$ or

$$L(1-r) = 0.$$

We assumed that $|r| < 1$, so $1 - r \neq 0$, and hence it follows that $L = 0$. |||

The proof just given is not valid. In fact, the argument shows that $\lim\{r^n\} = 0$ whenever $r \neq 1$, and this is certainly wrong when $r = 2$. The error comes in the first sentence, "Let $L = \lim\{r^{n-1}\}$". The argument works if the sequence $\{r^{n-1}\}$ or $\{r^n\}$ converges. We will now give a second (correct) proof of theorem 6.71.

Second Proof: Let r be a real number with $|r| < 1$. If $r = 0$, then $\{r^n\} = \{0\}$ is a constant sequence, and $\lim\{r^n\} = \lim\{0\} = 0$. Hence the theorem holds when $r = 0$, and we may assume that $r \neq 0$. Let ϵ be a generic positive number, If $n \in \mathbf{Z}^+$ we have

$$(|r^n - 0| < \epsilon) \iff (|r|^n < \epsilon) \iff (n\ln(|r|) < \ln(\epsilon))\,.$$

Now since $|r| < 1$, we know that $\ln(|r|) < 0$ and hence

$$(n\ln(|r|) < \ln(\epsilon)) \iff \left(n > \frac{\ln(\epsilon)}{\ln(|r|)}\right).$$

By the Archimedean property, there is some positive integer $N(\epsilon)$ such that $N(\epsilon) > \dfrac{\ln(\epsilon)}{\ln(|r|)}$. Then for all n in $\mathbf{Z}^+$

$$(n \geq N(\epsilon)) \implies \left(n > \frac{\ln(\epsilon)}{\ln(|r|)}\right) \implies (|r^n - 0| < \epsilon).$$

Hence $\lim\{r^n\} = 0$. |||

6.72 Exercise. Why was it necessary to make $r = 0$ a special case in the Second Proof above?

6.6 Geometric Series

6.73 Theorem (Geometric series) *Let r be a real number such that $|r| < 1$. Then*

$$\{\sum_{i=1}^{n} r^{i-1}\} \to \frac{1}{1-r} \tag{6.74}$$

Equation (6.74) is often written in the form

$$\sum_{i=1}^{\infty} r^{i-1} = \frac{1}{1-r} \quad or \quad \sum_{i=0}^{\infty} r^{i} = \frac{1}{1-r}$$

Proof: Let r be a real number such that $|r| < 1$, and for all $n \in \mathbf{Z}^+$ let

$$f(n) = \sum_{i=1}^{n} r^{i-1}.$$

Then by theorem 2.22 we have

$$f(n) = \frac{1-r^n}{1-r},$$

and hence

$$\begin{aligned}\lim\{f(n)\} &= \lim\left\{\frac{1-r^n}{1-r}\right\} \\ &= \frac{1}{1-r}\lim\{(1-r^n)\} \\ &= \frac{1}{1-r}(1-\lim\{r^n\})\end{aligned}$$

Hence by the nth power theorem

$$\lim\{f(n)\} = \frac{1}{1-r}(1-0) = \frac{1}{1-r}. \ \|$$

6.75 Exercise. Find the error in the following argument. Let R be a real number with $R \neq 1$, and for n in $\mathbf{Z}^+$, let

$$a_n = 1 + R + R^2 + \cdots + R^{n-1}.$$

Let $L = \lim\{a_n\}$. Then, by the translation rule

$$\begin{aligned}L = \lim\{a_{n+1}\} &= \lim\{1 + R + \cdots + R^n\} \\ &= \lim\{1 + R(1 + \cdots + R^{n-1})\} = \lim\{1 + Ra_n\}.\end{aligned}$$

Thus by the sum rule and product rule,

$$\begin{aligned}L &= \lim\{1\} + \lim\{Ra_n\} \\ &= 1 + R\lim\{a_n\} = 1 + RL.\end{aligned}$$

Now

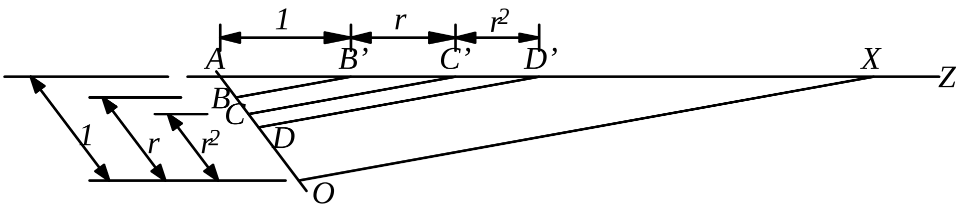
$$L = 1 + RL \implies L(1-R) = 1 \implies L = \frac{1}{1-R}.$$

Hence we have shown that

$$\lim\{1 + R + R^2 + \cdots + R^{n-1}\} = \frac{1}{1-R}$$

for all $R \in \mathbf{R} \setminus \{1\}$. (This sort of argument, *and the conclusion* were regarded as correct in the eighteenth century. At that time the argument perhaps *was* correct, because the definitions in use were not the same as ours.)

"The clearest early account of the summation of geometric series" [6, page 136] was given by Grégoire de Saint-Vincent in 1647. Grégoire's argument is roughly as follows:



On the line AZ mark off points B', C', D' etc. such that

$$AB' = 1, \;\; B'C' = r, \;\; C'D' = r^2, \;\; D'E' = r^3 \cdots$$

On a different line through A mark off points O, B, C, D etc. such that

$$OA = 1, \;\; OB = r, \;\; OC = r^2, \;\; OD = r^3 \cdots$$

Then

$$\begin{aligned}
\frac{AB'}{AB} &= \frac{1}{1-r}. \\
\frac{B'C'}{BC} &= \frac{r}{r-r^2} = \frac{1}{1-r}. \\
\frac{C'D'}{CD} &= \frac{r^2}{r^2-r^3} = \frac{1}{1-r}. \\
&\textit{etc.}
\end{aligned}$$

Now I use the fact that

$$\frac{a}{b} = \frac{c}{d} \Longrightarrow \frac{a+c}{b+d} = \frac{a}{b}, \tag{6.76}$$

(see exercise 6.78), to say that

$$\begin{aligned}
\frac{AC'}{AC} &= \frac{AB'+B'C'}{AB+BC} = \frac{AB'}{AB} = \frac{1}{1-r} \\
\frac{AD'}{AD} &= \frac{AC'+C'D'}{AC+CD} = \frac{AC'}{AC} = \frac{1}{1-r} \\
\frac{AE'}{AE} &= \frac{AD'+D'E'}{AD+DE} = \frac{AD'}{AD} = \frac{1}{1-r} \\
&\textit{etc.}
\end{aligned}$$

It follows that the triangles BAB', CAC', DAD', etc. are all mutually similar, so the lines BB', CC', DD' etc. are all parallel. Draw a line through O parallel to BB' and intersecting AZ at X. I claim that

$$AB' + B'C' + C'D' + D'E' + etc. = AX. \tag{6.77}$$

It is clear that any finite sum is smaller that AX, and by taking enough terms in the sequence $A, B, \cdots N$ we can make ON arbitrarily small. Then XN' is arbitrarily small, i.e. the finite sums AN' can be made as close to AX as we please. By similar triangles,

$$\frac{1}{1-r} = \frac{AB'}{AB} = \frac{AX}{AO} = \frac{AX}{1}$$

so, equation (6.77) says

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

6.78 Exercise. Prove the assertion (6.76).

6.79 Exercise.

a) Find $\lim\left\{1 + \left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \cdots + \left(\frac{9}{10}\right)^{n-1}\right\}$.

b) Find $\lim\left\{1 - \left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)^2 - \left(\frac{9}{10}\right)^3 + \cdots + \left(-\frac{9}{10}\right)^{n-1}\right\}$.

c) For each n in $\mathbf{Z}^+$ let

$$a_n = \sum_{j=0}^{\infty}\left(-\frac{n}{n+1}\right)^j,$$

(in part (b) you calculated a_9). Find a formula for a_n, and then find $\lim\{a_n\}$.

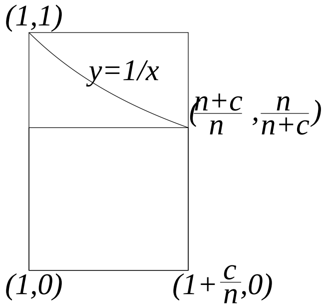
d) Show that

$$\lim\left\{\sum_{j=0}^{\infty}\left(-\frac{n}{n+1}\right)^j\right\} \neq \sum_{j=0}^{\infty}\lim\left\{\left(-\frac{n}{n+1}\right)^j\right\} \tag{6.80}$$

(Thus it is not necessarily true that the limit of an infinite sum is the infinite sum of the limits. The left side of (6.80) was calculated in part c. The right side is $\sum_{j=0}^{\infty} b_j$, where $b_j = \lim\left\{\left(-\frac{n}{n+1}\right)^j\right\}$ depends on j, but not on n.)

6.7 Calculation of e

6.81 Example. We will calculate $\lim\left\{n\ln\left(1+\frac{c}{n}\right)\right\}$, where c is a positive number. Let $f(x)=\frac{1}{x}$. Then (see the figure)



$$B(1,1+\frac{c}{n}:0,\frac{n}{n+c})\subset S_1^{1+\frac{c}{n}}f\subset B(1,1+\frac{c}{n}:0,1)$$

and hence

$$\text{area}(B(1,1+\frac{c}{n}:0,\frac{n}{n+c}))\le \text{area}(S_1^{1+\frac{c}{n}}f)\le \text{area}(B(1,1+\frac{c}{n}:0,1)).$$

Thus

$$\frac{c}{n}\cdot\frac{n}{n+c}\le \ln(1+\frac{c}{n})\le\frac{c}{n},$$

i.e.

$$\frac{cn}{n+c}\le n\ln(1+\frac{c}{n})\le c. \tag{6.82}$$

Since

$$\lim\left\{\frac{cn}{n+c}\right\}=\lim\left\{\frac{c}{1+\frac{c}{n}}\right\}=c,$$

it follows from the squeezing rule that

$$\lim\left\{n\ln\left(1+\frac{c}{n}\right)\right\}=c. \tag{6.83}$$

Notice that in this example the squeezing rule has allowed us to prove the existence of a limit whose existence was not obvious.

6.84 Example. We will show that for all $c \in \mathbf{Q}^+$

$$\lim\left\{\left(1+\frac{c}{n}\right)^n\right\} = e^c. \tag{6.85}$$

Let $c \in \mathbf{Q}^+$, and let $n \in \mathbf{Z}^+$. Let

$$a_n = \left(1+\frac{c}{n}\right)^n.$$

By (6.82), we have

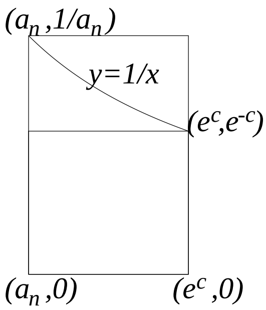
$$\ln(a_n) \leq c = c\ln(e) = \ln(e^c),$$

so

$$a_n \leq e^c \text{ for all } n \in \mathbf{Z}^+.$$

It follows from (6.83) that

$$\lim\{\ln(a_n)\} = c, \text{ or } \lim\{c - \ln(a_n)\} = 0. \tag{6.86}$$



From the picture, we see that

$$0 \leq B(a_n, e^c : 0, e^{-c}) \leq A_{a_n}^{e^c}\left[\frac{1}{t}\right],$$

i.e.

$$0 \leq e^{-c}(e^c - a_n) \leq \ln(e^c) - \ln(a_n) = c - \ln(a_n).$$

Hence

$$0 \leq e^c - a_n \leq e^c(c - \ln(a_n)), \text{ for all } n \in \mathbf{Z}^+.$$

By (6.86), we have

$$\lim\{e^c(c - \ln(a_n))\} = 0,$$

so by the squeezing rule, $\lim\{e^c - a_n\} = 0$, i.e.

$$\lim\{a_n\} = e^c.$$

This completes the proof of (6.85).

The reason we assumed c to be positive in the previous example was to guarantee that $(1+\frac{c}{n})$ has a logarithm. We could extend this proof to work for arbitrary $c \in \mathbf{Q}^+$, but we suggest an alternate proof for negative c in exercise 6.97.

6.87 Example (Numerical calculation of e) It follows from the last example that

$$\lim\left\{\left(1+\frac{1}{n}\right)^n\right\} = e.$$

I wrote a Maple procedure to calculate e by using this fact. The procedure `limcalc(n)` below calculates

$$\left(1+\frac{1}{100^n}\right)^{100^n},$$

and I have printed out the results for n = 1,2,. . . ,6.

```
> limcalc := n -> (1+ .01^n)^(100^n);
```

$$limcalc := n \to (1 + .01^n)^{(100^n)}$$

```
> limcalc(1);
```

2.704813829

```
> limcalc(2);
```

2.718145927

```
> limcalc(3);
```

2.718280469

```
> limcalc(4);
```

2.718281815

```
> limcalc(5);
```

1.

```
> limcalc(6);
```

1.

6.88 Exercise. From my computer calculations it appears that

$$\lim\left\{\left(1+\frac{1}{n}\right)^n\right\} = 1.$$

Explain what has gone wrong. What can I conclude about the value of e from my program?

6.89 Example. Actually, Maple is smart enough to find the limit, and does so with the commands below. The command `evalf` returns the decimal approximation of its argument.

```
> limit( (1+1/n)^n,n=infinity);
```

e

```
> evalf(%);
```

2.718281828

6.90 Entertainment ($\lim\{n^{\frac{1}{n}}\}$.) Find the limit of the sequence $\{n^{\frac{1}{n}}\}$, or else show that the sequence diverges.

6.91 Example (Compound interest.) The previous exercise has the following interpretation.

Suppose that A dollars is invested at r% annual interest, compounded n times a year. The value of the investment at any time t is calculated as follows:

Let $T = (1/n)$ year, and let A_n^k be the value of the investment at time kT. Then

$$\begin{aligned} A_n^0 &= A \\ A_n^1 &= A_n^0 + \frac{r}{100n}A_n^0 = (1+\frac{r}{100n})A \\ A_n^2 &= A_n^1 + \frac{r}{100n}A_n^1 = (1+\frac{r}{100n})^2 A \end{aligned} \tag{6.92}$$

and in general

$$A_n^k = A_n^{k-1} + \frac{r}{100n}A_n^{k-1} = (1+\frac{r}{100n})^k A. \tag{6.93}$$

The value of the investment does not change during the time interval $kT < t < (k+1)T$. For example, if V_n denotes the value of one dollar invested for one year at $r\%$ annual rate of interest with the interest compounded n times a year, then

$$V_n = A_n^n = \left(1 + \frac{r}{100n}\right)^n.$$

Thus it follows from our calculation that if one dollar is invested for one year at $r\%$ annual rate of interest, with the interest compounded "infinitely often" or "continuously", then the value of the investment at the end of the year will be

$$\lim\left\{\left(1 + \frac{r}{100n}\right)^n\right\} = e^{\frac{r}{100}} \text{ dollars.}$$

If the rate of interest is 100%, then the value of the investment is e dollars, and the investor should expect to get \$2.71 from the bank.

This example was considered by Jacob Bernoulli in 1685. Bernoulli was able to show that $\lim\left\{\left(1+\frac{1}{n}\right)^n\right\} < 3$.[8, pp94-97]

6.94 Exercise. Calculate the following limits.

a) $\lim\{(1+\frac{3}{n})^{2n}\}$.

b) $\lim\{(1+\frac{1}{3n})^{2n}\}$.

6.95 Exercise.

a) Use the formula for a finite geometric series,

$$1 + (1-a) + (1-a)^2 + \cdots + (1-a)^{n-1} = \frac{1-(1-a)^n}{1-(1-a)}$$

to show that

$$(1-a)^n \geq 1 - na \text{ whenever } 0 < a < 1. \tag{6.96}$$

b) Let $c \in \mathbf{R}^+$ Use inequality (6.96) to show that

$$\left(1 - \frac{c}{n^2}\right)^n \geq 1 - \frac{c}{n}$$

for all $n \in \mathbf{Z}^+$ such that $n > \sqrt{c}$.

c) Prove that $\lim\{(1 - \frac{c}{n^2})^n\} = 1$ for all $c \in \mathbf{R}^+$.

6.97 Exercise. Let $c \in \mathbf{Q}^+$. Use exercise 6.95 to show that

$$\lim\left\{\left(1 - \frac{c}{n}\right)^n\right\} = e^{-c}.$$

(Hence we have $\lim\{(1 + \frac{c}{n})^n\} = e^c$ for all $c \in \mathbf{Q}$.)
Hint: Note that $(1-z) = (\frac{1-z^2}{1+z})$ for all real numbers $z \neq -1$.