HEURISTIC DERIVATION OF STIRLING'S FORMULA BY ASYMPTOTICS

Euler's representation of the factorial function as the *gamma integral* is

$$n! = I_n = \int_{t=0}^{\infty} e^{-t} t^n \,\mathrm{d}t, \quad n = 0, 1, 2, \dots .$$

Indeed, it is easy to check that $I_0 = 1 = 0!$; and also an integration by parts gives $I_{n+1} = (n+1)I_n$ for $n \geq 0$, so now $I_n = n!$ for all n by induction.

The integrand is

$$e^{-t} t^n = e^{-t} e^{\ln(t^n)} = e^{-t} e^{n \ln(t)} = e^{-t + n\ln(t)},$$

and so we have

$$n! = \int_{t=0}^{\infty} e^{f_n(t)} \,\mathrm{d}t \quad \text{where } f_n(t) = -t + n\ln(t).$$

Here we take $f_n(0) = -\infty$, so that $e^{f_n(0)} = 0$.

If we plot the integrand $e^{f_n(t)}$ for $n = 1$, $n = 2$, $n = 5$, $n = 10$, etc., we see it come to resemble a bell-shaped curve centered at n though dampened to the left. So we think that asymptotically in n the integral becomes a Gaussian integral.

To quantify this idea, recall that $f_n(t) = -t + n\ln(t)$ and note that

- $f_n(n) = -n + n\ln(n) = -n + \ln(n^n)$,
- $f_n'(t) = -1 + n/t$ vanishes at $t = n$,
- $f_n''(t) = -n/t^2$ and so $f_n''(n) = -1/n$.

Thus the quadratic approximation of f_n about its maximizing input n is

$$f_n(t) \approx -n + \ln(n^n) - \frac{1}{2n}(t-n)^2.$$

This quadratic approximation gives an asymptotic approximation of the factorial,

$$n! \approx \int_{t=-\infty}^{\infty} e^{-n + \ln(n^n) - \frac{1}{2n}(t-n)^2} \,\mathrm{d}t \quad \text{for large } n.$$

Here we feel free to integrate from $-\infty$ to ∞ because the integral over the left half of the real axis will be small. Because $e^{-n+\ln(n^n)} = e^{-n}n^n = (n/e)^n$, this is

$$n! \approx (n/e)^n \int_{t=-\infty}^{\infty} e^{-\frac{1}{2n}(t-n)^2} \,\mathrm{d}t.$$

Recall the normalized Gaussian integral $\int_{x=-\infty}^{\infty} e^{-\pi x^2} \,\mathrm{d}x = 1$. In our approximating integral for $n!$, let $x = (t-n)/\sqrt{2\pi n}$, so that $-\frac{1}{2n}(t-n)^2 = -\pi x^2$ and $\mathrm{d}t = \sqrt{2\pi n}\,\mathrm{d}x$. This gives

$$n! \approx (n/e)^n \sqrt{2\pi n} \int_{x=-\infty}^{\infty} e^{-\pi x^2} \,\mathrm{d}x = (n/e)^n \sqrt{2\pi n}.$$

This is Stirling's formula.

The method used here is a special case of *Watson's lemma* from c.1919, well after Stirling's formula, so it is antihistorical.