Mathematics 361: Number Theory
Assignment B

Reading: Ireland and Rosen, Chapter 2 (including the exercises) and into Chapter 3

Problems:

Even perfect numbers (nobody knows if there are any odd ones).

1. The *sum of divisors* arithmetic function $\sigma(n) = \sum_{0<d|n} d$ is introduced in Ireland and Rosen, and $\sigma(p^a) = (p^{a+1} - 1)/(p - 1)$ by the finite geometric sum formula, and σ is multiplicative. A positive integer is called *perfect* if it is the sum of its proper positive divisors, i.e., if $\sigma(n) = 2n$.

(a) Show that if $2^p - 1$ is prime (forcing p to be prime) then $2^{p-1}(2^p - 1)$ is perfect.

(b) If m is even and perfect, show that m takes the form $m = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime. (Write $m = 2^{p-1}t$ where $p \geq 2$ and t is odd and we don't yet know whether p is prime. Show that $\sigma(t) = 2^p r$ where r is odd and $t = (2^p - 1)r$. Note that r and t are distinct factors of t. Use this to show that $r = 1$ and $2^p - 1$ is prime.)

(c) Where does the argument in (b) break down if $p = 1$? That is, why can't we argue as in (b) to show that there are no odd perfect numbers?

2. Work Ireland and Rosen exercises 1.30, some of 1.32–1.38.

3. Prove yet again that there are infinitely many primes by working Ireland and Rosen exercise 2.3 or exercises 2.4–2.5 or exercises 2.6–2.8.

4. Get some practice with arithmetic functions by working a selection from Ireland and Rosen exercises 2.9–2.21.

5. Acquire some initial familiarity with the Riemann zeta function of analytic number theory by reading and perhaps working Ireland and Rosen exercises 2.25–2.26. Also, 2.27 is a nice variant of Euler's argument; to get going on it, note that every positive integer n factors uniquely as $n = m^2 r$ with r squarefree.