A BRIEF INTRODUCTION TO THE LIE BRACKET

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1. Introduction

Several (very good) books on Lie groups and algebras begin with absolutely compelling introductions. *Lie groups are, broadly speaking, the study of symmetry. They are ubiquitous in modern mathematics and physics.*

Eventually, it is revealed that in order to study the properties of Lie groups, one employs Lie algebras. A definition is provided, and it is one that I did not understand. Lie algebras, apparently, are closely related to - and sometimes defined with - something called Lie *brackets*, which are fancy looking and strange sounding.

The goal of this paper is to motivate the definition of a Lie algebra and the Lie bracket by studying *homomorphisms* of Lie groups and thinking in terms of *group actions*. We hope that by the end of the paper the reader will feel that the definition of a Lie algebra is something natural, and worthy of the lofty introductions that the subject often receives.

To this end, we have not shied away from employing results from manifold theory when necessary. We hope that Appendix B will provide sufficient background to follow the main narrative. Similarly, we make frequent use of multilinear algebra, but hope that Appendix A provides enough vocabulary to follow the text.

2. Representations of Finite Groups

We begin by discussing representations of groups, since we will eventually be interested in representations of Lie groups. The discussion begins with some definitions and properties, and concludes with two basic theorems. The section follows Lecture One in [1].

Definition 2.1 (Representation)**.** *A representation of a finite group G on a finite dimensional complex vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ of G to the group of automorphisms of V.*

The definition tells us that for every element $g \in G$ there is a group homomorphism $\rho(g)$ that acts on the vector space V. In this way, we can think of a representation as a linear *action* of a group on a vector space.

Thinking of representations as actions on vector spaces should help motivate our referential notation, which will be somewhat abusive. We will often call V itself a representation of G, and we will write $g \cdot v$ or merely gv to mean $\rho(g)(v)$. We call the dimension of V the *degree* of ρ. We say that the map ρ *gives V the structure of a G-module.*

A map φ between two representations V and W of G is a vector space map $\varphi : V \to W$ such that $g\varphi = \varphi g, \forall g \in G$. That is to say:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

commutes for all $g \in G$.

Definition 2.2 (Subrepresentation)**.** *A subrepresentation of a representation V is a vector subspace W of V which is invriant under G. That is,*

$$gw \in W \quad \forall w \in W,\ g \in G$$

So a subrepresentation is just an invariant subspace of G that is mapped to itself by g.

Definition 2.3 (Irreducible representation)**.** *A representation V is called irreducible if there is no proper nonzero invariant subspace W of V.*

If V and W are representations, the direct sum $V \oplus W$ is a representation via

$$g(v \oplus w) = gv \oplus gw$$

Likewise, the tensor product $V \otimes W$ is a representation via

$$g(v \otimes w) = gv \otimes gw$$

Since the tensor product is a representation, the n-th tensor power $V^{\otimes n}$ is a representation as well. The exterior and symmetric powers of V are subrepresentations of the $V^{\otimes n}$ representation[1].

So representations may be constructed out of other representations via linear algebra operations, the simplest of which is the direct sum. It is natural to focus on representations that are atomic with respect to the direct sum. That is:

Definition 2.4 (Irreducible)**.** *We call a representation V of a group G irreducible iff V has no nontrivial invariant subspaces.*

Theorem 2.5 (Maschke)**.** *Every representation of a finite group having positive dimension is completely reducible*

Given this theorem, it is natural to consider the extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique. The issue is addressed by *Shur's Lemma*:

Lemma 2.6 (Shur)**.** *If V and W are irreducible representations of G and $\varphi : V \to W$ is a G-module homomorphism, then either φ is an isomorphism or $\varphi = 0$.*

Proof. Since V is irreducible and $\mathrm{Ker}\varphi$ is an invariant subspace of V, $\mathrm{Ker}\varphi = 0$ or $\mathrm{Ker}\varphi = V$. Likewise, W is irreducible and therefore $\mathrm{Im}\,(\varphi) = 0$ or $\mathrm{Im}\,(\varphi) = W$. If $\ker\varphi = V$ or $\mathrm{Im}\,(\varphi) = 0$, then φ is the zero map. If $\ker\varphi = 0$ and $\mathrm{Im}\,(\varphi) = W$, then we have an isomorphism. This completes the proof. $\square$

[1] If the reason for this is not clear, please see Appendix A

3. Lie Groups

With the idea of a representation behind us, we begin this short section on Lie groups. The goal of the section is simply to provide a good idea of what a Lie group *is*, and the idea of a group as a manifold is used from the beginning. If the reader is unfamiliar with manifolds, now is a good time to consult Appendix B.

Definition 3.1. *A lie group G is a set endowed with the structure of a group and a C^∞ manifold such that the group operations*

$$\mu : G \times G \quad \mu(a, b) = ab$$

$$\iota : G \to G, \quad \iota(a) = a^{-1}$$

are C^∞ mappings.

An example may help clarify the definition.

Example 3.2 ($\mathrm{GL}(n, \mathbf{R})$). The general linear group of $n \times n$ matrices is an open subset of $\mathbf{R}^{n\times n}$ and therefore a manifold.[2]

Matrix multiplication

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad A, B \in \mathrm{GL}(n, \mathbf{R})$$

is a polynomial map in each coordinate and therefore C^∞. By Cramer's rule, we have for inverse map:

$$(A^{-1})_{ij} = \frac{1}{\det A} \mathrm{Adj} A \qquad \mathrm{Adj} A := \text{transpose of the matrix of cofactors}$$

The coordinates of $\mathrm{Adj}A$ are polynomial functions in the coordinates of A, so the inverse map is C^∞ function provided that $\det A \neq 0$, which is guaranteed by the definition of $\mathrm{GL}(n, \mathbf{R})$. Thus, $\mathrm{GL}(n, \mathbf{R})$ is a Lie group.

We can also think of $\mathrm{GL}_n\mathbf{R}$ as the group of automorphisms of an n-dimensional real vector space V, in which case we write $\mathrm{GL}(V)$ or simply $\mathrm{Aut}(V)$ instead of $\mathrm{GL}_n\mathbf{R}$. Of course we define the *representation* of a Lie group to be a smooth group homomorphism $\rho : G \to \mathrm{GL}(V)$.

4. Lie Algebras

We want to study these representations of Lie groups. Naturally, our investigation is going to center around homomorphisms. We begin with a motivating fact.

Fact 4.1. Let G be a connected Lie group, and $U \subset G$ be a neighborhood of the identity. U generates G.

This means that any map $\rho : G \to H$ between connected Lie groups G, H may be determined by its germ at $e \in G$. In fact, we will see that

Fact 4.2. Let G and H be Lie groups with G connected. A map[3] $\rho : G \to H$ is uniquely determined by its differential $d\rho_e : T_eG \to T_eH$ at the identity.

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[2]If this is not clear, see Appendix B.

[3]We will say *map* or *morphism* to mean a C^∞ group homomorphism.

Given this fact, the question arises: *which* maps between these two vector spaces actually arise as differentials of group homomorphisms? The answer, as we will see by the end of this section is as follows:

Fact 4.3. A linear map $T_eG \to T_eH$ is the differential of a homomorphism $\rho : G \to H$ iff it preserves the bilinear structure $d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)]$

The brackets are called *Lie brackets*, and are fundamental to the definition of Lie Algebras. The condition is (as of yet) unmotivated, poorly defined, and mysterious; by the end of the section, however, it should feel inevitable.

We begin by noticing that a Lie group homomorphism *respects* the *action* of a group on itself. First, a definition.

A Lie group homomorphism $\rho : G \to H$ is a C^∞ map that satisfies

$$\rho(gh) = \rho(g) \cdot \rho(h)$$

for all $g, h \in G$. This can be represented diagrammatically as follows:

$$\begin{tikzcd} G \times G \arrow[r, "\rho"] \arrow[d, "\cdot"'] & H \times H \arrow[d, "\cdot"] \\ G \arrow[r, "\rho"'] & H \end{tikzcd}$$

We can of course view $G \times G$ as a group action. Merely let one of the Gs be a *set* and the group acts on itself by either left or right multiplication, accordingly. We therefore say that a homomorphism respects the *action* of a group on itself by left or right multiplication. We will make the diagram above a little fancier by explicitly defining a function for this group action.

Let

$$m_g : G \to G$$

be the differentiable map given by multiplication by any $g \in G$. We then say that a Lie group map $\rho : G \to H$ will be a homomorphism if the diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "m_g"'] & H \arrow[d, "m_{\rho(g)}"] \\ G \arrow[r, "\rho"'] & H \end{tikzcd}$$

commutes.

The diagram asserts that it doesn't matter if you perform the group action before or after the mapping is applied, which is of course what the previous diagram showed as well! Since we are working towards the *differential* of a group homomorphism, defining an explicit mapping was a natural thing to do. The natural question to ask now, then, is: should we take the differential?

Unfortunately, no, because it's not clear where we would take the differential since m_g leaves no fixed points. Happily, this situation is easily resolved by considering the automorphisms of G given by conjugation. We define

$$\Psi_g : G \to G, \quad h \mapsto g \cdot h \cdot g^{-1}$$

for $g, h \in G$.

Of course Ψ_g fixes the identity element $e \in G$, *and* is an action respected by homomorphisms, which is to say that the diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\Psi_g"'] & H \arrow[d, "\Psi_{\rho(g)}"] \\ G \arrow[r, "\rho"'] & H \end{tikzcd}$$

commutes.

So we are now in a position to look at the differential of the map Ψ at e, which will give

$$(d\Psi_g)_e : T_eG \to T_eG$$

We denote this map $\mathrm{Ad}(g)$, and call it the *adjoint* representation of the group. Since Ψ_g was a map from G to the automorphisms of G (given by conjugation) we think of Ad as a map from g to the automorphisms of T_eG, or as a *representation* of the group G on its tangent space T_eG:

$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG)$$

Ad gives the commutative diagram

$$\begin{tikzcd} T_eG \arrow[r, "(d\rho)_e"] \arrow[d, "\mathrm{Ad}(g)"'] & T_eH \arrow[d, "\mathrm{Ad}(\rho(g))"] \\ T_eG \arrow[r, "(d\rho)_e"'] & T_eH \end{tikzcd}$$

which asserts that a homomorphism respects the adjoint action of a group on its tangent space at the identity, which is to say that

$$d\rho(\mathrm{Ad}(g)(v)) = \mathrm{Ad}(\rho(g))(d\rho(v))$$

for $g \in G$, $v \in T_eG$.

This is just a little bit of a problem, since the adjoint representation still involves the map ρ on the group G. We want a condition that is *purely* on the differential.

In order to achieve this condition, we will have to modify the Ad function. Specifically, we will take the differential of the map Ad. The group $\mathrm{Aut}(T_eG)$ is an open subset of the vector space $\mathrm{End}(T_eG)$, and therefore taking the differential of the map

$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG)$$

at the identity e gives

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$$

Like Ad before it, the map ad gives a map from the tangent space to itself and thus the commutative diagram

$$\begin{tikzcd} T_eG \arrow[r, "(d\rho)_e"] \arrow[d, "\mathrm{ad}(v)"'] & T_eH \arrow[d, "\mathrm{ad}(d\rho(v))"] \\ T_eG \arrow[r, "(d\rho)_e"'] & T_eH \end{tikzcd}$$

Notice how things have changed. The Ad operator was an adjoint action of the *group* on its tangent space at the identity, the ad operator is an adjoint action of the *tangent space* to the group on its tangent space. Thus, the diagram asserts that the following equality holds for tangent vectors $X, Y \in T_eG$

$$d\rho_e(\mathrm{ad}(X)(Y)) = \mathrm{ad}(d\rho_e(X))(d\rho_e(Y))$$

which is a condition purely on the differential, as desired.

We can view the image $\mathrm{ad}(X)(Y)$ of a tangent vector Y under the map $\mathrm{ad}(X)$ as a function of the two variables X and Y which gives a bilinear map

$$T_eG \times T_eG \to T_eG.$$

We use the notation $[\ ,\ ]$ to denote this map, and therefore write

$$[X, Y] = \mathrm{ad}(X)(Y)$$

for tangent vectors X, Y to G at the identity e. In this new notation, the diagram above asserts that the equality

$$d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)]$$

holds.

Let's turn to the general linear group in order to make things explicit. Its tangent space is the space of endomorphisms of R^n, and therefore differentiation is just differentiation of matrices. For tangent vectors X, Y to $GL_n\mathbf{R}$ at e, let $\gamma : I \to G$ be an arc with $\gamma(0) = e$ and $\gamma'(0) = X$. We calculate:

$$[X, Y] = \mathrm{ad}(X)(Y) = \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}(\gamma(t))(Y)).$$

Note that $\mathrm{Ad}(\gamma(t))(Y) = \gamma(t)Y\gamma(t)^{-1}$ and apply the product rule to obtain

$$\begin{aligned} &= \gamma(0) \cdot Y \cdot \gamma(0) + \gamma(0) \cdot Y \cdot (-\gamma(0)^{-1} \cdot \gamma'(0) \cdot \gamma(0)^{-1}) \\ &= X \cdot Y - Y \cdot X \end{aligned}$$

So the bracket operation coincides with the commutator. Two properties fall out immediately.

(1) The bracket is skew-symmetric:

$$[X, Y] = -[Y, X].$$

(2) The bracket sasifies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

This leads us, finally, to our (natural!) definition of a Lie algebra:

Definition 4.4. *A lie algebra $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map*

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that satisfies the Jacobi identity.

5. Concluding Remarks

The reader may lament that we never *justified* our motivating facts, and this is certainly true. In order to prove the assertions that were made at the outset of the previous section and guided our discussion of Lie algebras, we need to discuss the *exponential mapping*, which connects Lie groups and algebras. Certainly, this is the natural place to proceed next. We consider the exponential mapping to be outside the scope of this paper, however, because in order to do it justice in context a fair bit of additional manifold theory would need to be introduced (namely, the idea of a *subgroup* of a Lie algebra, and therefore the *immersed submanifold*), which is not something that is necessarily desirable given the amount of new vocabulary already introduced. We therefore refer the reader to the (excellent) Section 8.3 of [1] for a well motivated discussion of the exponential mapping[4].

Appendix A. Multilinear Algebra

The purpose of this section is to establish some essential properties of tensor products, and exterior and symmetric powers. The discussion is essentially an exposition of Appendix B in [1], with exposition aided significantly by [4].

We will assume throughout this section that every field has characteristic 0 and all vector spaces are finite dimensional.

The first thing to do is define a *bilinear mapping* which is simply an extension of the notion of linear mapping to two variables

Definition A.1 (Bilinear Mapping)**.** *We say that a mapping*

$$\varphi : E \times F \to G$$

is bilinear if it satisfies

$$\varphi(\lambda x_1 + \mu x_2, y) = \lambda\varphi(x_1, y) + \mu\varphi(x_2, y)$$

$$\varphi(x, \lambda y_1 + \mu y_2) = \lambda\varphi(x, y_1) + \mu\varphi(x, y_2)$$

where E, F, G *are linear spaces over some field* Γ*, and* $x_n \in E,\ y_n \in F$ *and* $\lambda, \mu \in \Gamma$.

A bilinear mapping is a two-space instance of a (more general) *multilinear mapping*. A multilinear mapping, of course, is linear in each of its variables while the others are held constant. From now on we will use the idea of a multilinear and bilinear mappings without comment.

Definition A.2 (Tensor Product)**.** *Let* $\varphi : E \times F \to G$ *be a bilinear mapping. The pair* (G, φ) *is called a* TENSOR PRODUCT *for* E *and* F *if, given any bilinear mapping* $\psi : E \times F \to H$ *there exists a unique linear mapping* $f : G \to H$ *such that* $\psi = f \circ \varphi$.

Diagrammatically, the definition states that any diagram of the form

$$\begin{tikzcd} E \times F \arrow[r, "\psi"] \arrow[d, "\varphi"'] & H \\ G & \end{tikzcd}$$

[4]Alternatively, I also have a (rough) writeup available upon request.

can always be completed by some unique f such that

$$\begin{tikzcd} E \times F \arrow[r, "\psi"] \arrow[d, "\varphi"'] & H \\ G \arrow[ur, dotted, "f"] & \end{tikzcd}$$

commutes.

Note that a tensor product is a *pair* (G, φ). That is, a tensor product is a vector space equipped with a bilinear map. We write the vector space $G \equiv E \otimes F$, and the canonical bilinear map $\varphi(x, y) \equiv x \otimes y$. We will often refer to a tensor product by its vector space $E \otimes F$, but the reader should keep in mind that we are implicitly asserting the existence of the canonical bilinear map as well.

In this notation, the definition given above says that a tensor product $E \otimes F$ satisfies the diagram:

$$\begin{tikzcd} E \times F \arrow[r, "\psi"] \arrow[d, "\otimes"'] & H \\ E \otimes F \arrow[ur, dotted, "f"] & \end{tikzcd}$$

Where

$$E \times F \to E \otimes F, \quad x \times y \mapsto x \otimes y$$

It follows that if $\{\alpha_i\}$ and $\{\beta_j\}$ are bases for E and F, the elements $\{\alpha_i \otimes \beta_j\}$ form a basis for the tensor product $E \otimes F$. We therefore have $\dim(E \otimes F) = \dim E \cdot \dim F$.

The construction of tensor products is commutative:

$$E \otimes F \cong F \otimes E$$

distributive:

$$(E_1 \oplus E_2) \otimes F \cong (E_1 \otimes F) \oplus (E_2 \otimes F),$$

and associative:

$$(D \otimes E) \otimes F \cong D \otimes (E \otimes F) \cong D \otimes E \otimes F$$

We denote a *tensor power* of a fixed space V:

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$$

This is sufficient discussion of the tensor product for the time being. We move toward external powers by introducing the idea of a *skew symmetric mapping.*

Definition A.3 (Skew symmetric mapping)**.** *Let φ be a p-linear mapping of $\underbrace{E \times \cdots \times E}_{p}$ into F. Any permutation σ will determine some other p-linear mapping $\sigma\varphi$ given by*

$$\sigma\varphi(x, \cdots, x_p) = \varphi(x_{\sigma(1)}, \cdots, x_{\sigma(p)})$$

A p-linear mapping is called skew symmetric iff

$$\sigma\varphi = \begin{cases} \varphi & \text{if the permutation is even} \\ -\varphi & \text{if the permutation is odd} \end{cases}$$

A number of equivalent properties of skew-symmetric mappings φ fall out easily:

(1) For any permutation σ,
$$\tau\varphi = -\varphi$$
since transposition is an odd permutation.
(2) φ is *alternating*. That is,
$$\varphi(x_1, \cdots, x_p) = 0 \iff x_i = x_j \text{ for at least one pair } i \neq j.$$
(3) For any permutation σ,
$$\varphi(x_{\sigma(1)}, \cdots, x_{\sigma(p)}) = \operatorname{sgn}(\sigma)\varphi(x_1, \cdots x_p)$$

Definition A.4 (Symmetric mapping)**.** *Let E and F be vector spaces and let $\varphi : \underbrace{E \times \cdots \times E}_{p} \to F$ be a p-linear mapping. Then φ is called symmetric if $\varphi = \sigma\varphi$ for every permutation σ.*

The idea of skew-symmetric and symmetric mappings will play a prominent role as we move to exterior and symmetric powers.

Definition A.5 (Exterior powers of a vector space)**.** *Let E be an arbitrary vector space and $p \geq 2$ be an integer. A vector space $\bigwedge^p E$ together with a p-linear skew-symmetric map*
$$\wedge^p : E \times \cdots \times E \to \wedge^p E,$$
is called a p-th exterior power of E if, given any skew symmetric p-linear mapping $\psi : E \times \cdots \times E \to F$, there exists a unique linear mapping $f : \wedge^p E \to F$ such that $\psi = f \circ \wedge^p$.

Diagrammatically, the definition gives a commutative diagram

$$\begin{tikzcd}
E \times \cdots \times E \arrow[r, "\psi"] \arrow[d, "\wedge^p"'] & F \\
\wedge^p E \arrow[ur, dotted, "f"'] &
\end{tikzcd}$$

where
$$E \times \cdots \times E \to \wedge^p E, \qquad x_1 \times \cdots \times x_p \mapsto x_1 \wedge \cdots \wedge x_p$$

As with the tensor product, we will denote the exterior powers of a vector space E by $\wedge^p E$, and leave the existence of a p-linear skew-symmetric multilinear map implicit.

The exterior product can be constructed by considering the space $E^{\otimes p}$ and quotienting away the subspace consisting of all $x_1 \otimes \cdots \otimes x_p$ with two vectors equal. More formally:
$$\wedge^p E = E^{\otimes p}/T \quad\quad T = \{x_1 \otimes \cdots \otimes x_p : x_i = x_j \text{ for some } i \neq j\}$$
We denote this projection:
$$\pi : E^{\otimes p} \to \wedge^p E$$
giving,
$$\pi(x_1 \otimes \cdots \otimes x_p) = x_1 \wedge \cdots \wedge x_n$$

We state without proof that if $\{e_i\}$ is a basis for V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : i_1 < \cdots < i_p\}$ is a basis for $\wedge^p E$.

We now move to symmetric powers, whose definition should come as no surprise.

Definition A.6 (Symmetric power)**.** *Let E be a vector space and $p \geq 2$ be an integer. A vector space S^pE together with a p-linear mapping: $S^n : E \times \cdots \times E \to S^pE$ is called a p-th symmetric power of E if for any symmetric mapping $\psi : E \times \cdots \times E \to F$, there exists a unique linear mapping $f : S^pE \to F$ such that $\psi = f \circ S^p$*

The definition gives the commutative diagram

$$\begin{tikzcd} E \times \cdots \times E \arrow[r, "\psi"] \arrow[d, "S^p"'] & F \\ S^pE \arrow[ur, dotted, "f"'] & \end{tikzcd}$$

where

$$E \times \cdots \times E \to S^pE, \qquad x_1 \times \cdots \times x_p \mapsto x_1 \cdot \cdots \cdot x_p$$

As is now standard, we leave the $p-$linear mapping implicit and denote the symmetric powers of E by S^pE.

The symmetric power can be constructed as the quotient space of $E^{\otimes p}$ by the linear span of expressions of the form

$$x_1 \otimes \cdots \otimes x_p - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)}$$

If $\{e_i\}$ is a basis for E, then

$$\{e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_p} : i_1 \leq i_2 \leq \cdots \leq i_p\}$$

is a basis for S^pE.

Appendix B. Manifolds

The purpose of this section is to develop the ideas of *tangent space* and *differential* which play prominent roles in our discussion of Lie algebras.

A topological manifold is just a topological space that looks locally like a piece of $\mathbf{R}^n$. More formally:

Definition B.1 (Topological manifold)**.** *A topological manifold of dimension n is a Hausdorff, second countable topological space M that is locally homeomorphic to an open subset of $\mathbf{R}^n$.*

The definition is asserting that for any point $m \in M$, there is some open neighborhood U of m that is homeomorphic to an open neighborhood V of $\mathbf{R}^n$. This homeomorphism

$$\varphi : U \to V$$

is called a *chart*. We call a collection of charts $\{U_\alpha, \varphi_\alpha\}$ that cover M an *atlas*.

The simplest example of a manifold is, of course, Euclidean space itself. $\mathbf{R}^n$ is covered by a single chart, $(\mathbf{R}^n, 1_{\mathbf{R}^n})$ where $1_{\mathbf{R}^n} : \mathbf{R}^n \to \mathbf{R}^n$ denotes the identity map. We will only be interested in a *smooth manifolds*, which are manifold having a *maximal* atlas and the property that any two charts of the manifold have *smooth transition functions* between them.

Definition B.2 (Smooth manifold)**.** *A smooth (or C^∞) manifold is a topological manifold M along with an atlas such that:*

$$M = \cup_\alpha U_\alpha$$

And for all α, β, with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map

$$\varphi_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth on $\mathbf{R}^n$.

We will now turn to a very important example. We begin with a familiar fact from linear algebra:

Fact B.3. The set of $n \times n$ real matrices forms a real vector space of dimension n^2. That is:

$$\mathbf{R}^{n\times n} \cong \mathbf{R}^{n^2}$$

where $\mathbf{R}^{n\times n}$ denotes the vector space of all n by n matrices.

Example B.4 ($GL(n, \mathbf{R})$)*.* The general linear group over $\mathbf{R}$, $\mathrm{GL}(n, \mathbf{R})$, is a smooth manifold. Recall that

$$\mathrm{GL}(n, \mathbf{R}) = \{A \in \mathbf{R}^{n\times n} | \det A \neq 0\}$$

We know that the determinant

$$\det : \mathbf{R}^{n\times n} \to \mathbf{R}$$

is continuous, and therefore that $\det^{-1}\{0\}$ is a *closed* subset of $\mathbf{R}^{n\times n}$. It follows that $\det^{-1}\{\mathbf{R}/\{0\}\}$ is an *open* subset of $\mathbf{R}^{n\times n}$. Given fact B.3, we see that $\mathrm{GL}(n, \mathbf{R})$ satisfies definition B.1 and thus is a manifold.

We will now turn to a discussion of tangent vectors and spaces, which is essential to understand as we proceed to Lie Algebras.

First, we define a germ.

Definition B.5 (Germ)**.** *Let two real-valued functions, each defined and differentiable in some neighborhood of a point p of M, be called equivalent if they agree in a neighborhood of p. The equivalence classes are called germs of differentiable functions on M at p. We denote the set of these germs $C_p^\infty M$.*

It is straightforward to observe that $C_p^\infty M$ is a *ring* with the operations of addition and multiplication, and scalar multiplication by $n \in \mathbf{R}$ makes $C_p^\infty M$ an *algebra* on $\mathbf{R}$.

Definition B.6 (Tangent vector)**.** *We define a tangent vector v to be a derivation of the ring of germs $C_p^\infty M$*

$$v : C_p^\infty M \to \mathbf{R}$$

that satisfies the product rule

$$v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g) \quad \forall f, g \in C_p^\infty M$$

We denote the vector space of these derivations $T_p(M)$ and call it the *tangent space* to M at p.

We will now discuss the *differential*, which is the local linear approximation of a differentiable map between manifolds.

Definition B.7 (Differential)**.** *Let $f : M \to N$ be a differentiable map between manifolds, $p \in M$, $v \in T_pM$, $\varphi \in C_{f(p)}^\infty$. The linear map*

$$df_p : T_pM \to T_{f(p)}N,$$

is called the differential of f at the point p.

The differential has the properties that we would expect of it.

Remark B.8. The differential of the identity is the identity,

$$d\mathrm{Id}_p = \mathrm{Id}_{T_pM}$$

And the chain rule holds,

$$d(g \circ f)_p = dg_{f(p) \circ df_p}$$

for differentiable maps $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$.

References

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