PONTRJAGIN DUALITY

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Note we will be working only in cartesian closed categories where exponentials and finite products exist for all objects X, Y.

1. The Exponential Object

An important construction in category theory is the **exponential construction**. Simply, the exponential construction generalizes the ideas of a mapping set to a category theoretic universal structure.

Let $\mathcal{C}$ be a category. Let $X, B, Y \in \mathcal{C}$. Note that if we are given any map $f : X \times B \xrightarrow{f} Y$, then for each element $x \in X$, we can consider the following diagram,

$$\begin{tikzcd}
1 \times B \arrow[r, "\langle x, 1_B \rangle"] & X \times B \arrow[d, "\varphi"] \\
B \arrow[u, "\langle , 1_B \rangle"] \arrow[r, "\varphi_x"'] & Y
\end{tikzcd}$$

If the diagram commutes, we must have that $\varphi_x : B \to Y$ be the map defined such that,

$$\varphi_x(b) = \varphi(x, b)$$

Thus the simple map φ on the product $X \times B$ gives rise to a family of maps from B to Y that are parameterized by X.

If exponentials exist, one can furthur assume the following two properties,

(1) Every map $B \to Y$ occurs as φ_x for at least one $x \in X$.
(2) $\varphi_x = \varphi_y$ only if $x_1 = x_2$.

Thus we write Y^B instead of X and rename φ the function eval. Thus for any map $f : B \to Y$, let $\ulcorner f \urcorner : 1 \to Y^B$ be the unique element of Y^B guaranteed by properties 1. and 2. Then we know that for all f, b the following diagram commutes,

$$\begin{tikzcd}
1 \arrow[r, "b"] \arrow[d, "\langle \ulcorner f \urcorner, b \rangle"'] & B \arrow[d, "f"] \\
Y^B \times B \arrow[r, "\text{eval}"'] & Y
\end{tikzcd}$$

where we define eval $\langle \ulcorner f \urcorner, b \rangle = fb$. Thus we define the exponential object,

Definition 1.1 (Exponential Object)**.** *Given a category $\mathcal{C}$, for any two sets $B, Y \in \mathcal{C}$, define the exponential of B relative to Y as the 2-tuple $(Y^B, eval)$ where Y^B is a set in $\mathcal{C}$ and the eval is defined as a morphism from $Y^B \times B$ to Y such that for any X and $X \times B \xrightarrow{f} Y$, there is a unique morphism $\ulcorner f \urcorner : X \to Y^B$ such that eval$(\ulcorner f \urcorner \times 1_B) = f$.*

In the categories we will be working in, we can simplify this definition a little bit and say that given two objects $X, Y \in \mathcal{C}$ the exponential Y^X is the internal hom of X, Y, i.e. the hom set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ which is gauranteed to be an object of $\mathcal{C}$.

2. Functoriality of the Exponential Object

Let $\mathcal{C}$ be a category. Given a set V, define the contravariant dual functor $(\hat{\ })$: $\mathcal{C} \to \mathcal{C}$ by,

$$X \mapsto V^X$$
$$\mathrm{Hom}(X, Y) \mapsto \mathrm{Hom}(V^Y, V^X)$$

for $X, Y \in \mathcal{C}$. Note that given $f : X \to Y$ we will write $(\hat{\ })f$ as V^f and that given $\alpha \in \mathrm{Hom}(Y, V)$,

$$V^f(\alpha) = \alpha \circ f$$

3. Concrete Duality

Recall the category theoretic concept of "abstract duality" or "duality". Many concepts in category theory are defined such that pairs or "duals" of such concepts exist. Some examples of this include monomorphism/epimorphism, limit/colimit, etc. But even though this duality is often times useful, since it dumbly just reverses all arrows in the relevant diagrams, any specific interpretations of the diagrams in terms of specific sets and mappings has been lost.

On the other hand given a specific V, we can use the contravariant dual functor to transform any specific diagram,

$$X \rightleftarrows Y$$

produces a specific diagram on a larger function set in which all the arrows have been reversed,

$$V^X \leftrightarrows V^Y$$

but which satisifies all of the commutativites satisfied in the original diagram with the order of composition reversed. This technique using the dualizing functor is referred to as "conrete duality with respect to V" or "dualizing into V". Some important examples of the usage of this technique are,

(1) Stone Duality
(2) Gelfand-Naimark Duality
(3) Pontrjagin Duality

4. The Double Exponential Functor

Then we note that in certain cases we can reverse the concrete duality through the use of the contravariant double dual functor $(\hat{\ }\hat{\ })$ with respect to V, which maps,

$$X \mapsto (\hat{\ }\hat{\ })X$$
$$\mathrm{Hom}(X, Y) \mapsto \mathrm{Hom}(V^X, V)$$

where $f : X \to Y$ is defined by,

$$x \longmapsto \ulcorner \varphi \mapsto \varphi x \urcorner$$

The map $(\hat{\hat{\ }})$, sometimes called the Fourier transform or the Dirac delta, is a natural transformation since given any $X \xrightarrow{f} Y$, easily the following diagram is commutative,

$$\begin{array}{ccc} X & \xrightarrow{(\hat{\hat{\ }})} & V^{V^X} \\ \downarrow f & & \downarrow V^{V^f} \\ Y & \xrightarrow{(\hat{\hat{\ }})} & V^{V^Y} \end{array}$$

Even though naturality follows easily, there is no generalized proof that $(\hat{\hat{\ }})$ is a natural isomorphism. Instead it can only be proven in specific cases. In the following we prove the Pontrjagin Duality, which is the proof the $(\hat{\hat{\ }})$ is a natural isomorphism in the category of locally compact abelian groups.

5. Definitions and Planning

We begin by defining a locally compact topological set,

Definition 5.1 (Locally Compact Topological Set). *Given a topological set S, S is locally compact if and only if every point x of S has a neighborhood U such that $\overline{U}$ is compact. Any such U is called a compact neighborhood of x.*

Let **LCA** be the cartesian closed category of locally compact abelian groups. Since **LCA** is a cartesian closed category, we know that X^Y is defined for any $X, Y \in \mathbf{LCA}$.

Let $\mathbf{T}$ be the compact quotient group $\mathbf{R}/\mathbf{Z}$. Easily one can see that $\mathbf{T}$ is a member of **LCA** with the topology it inherits from $\mathbf{R}$. Thus given any $G \in \mathbf{LCA}$ we can consider the exponential of G relative to $\mathbf{T}$, $\mathbf{T}^G$. To ease our notation a little we will notate $\mathbf{T}^G$ as G^*.

Since $\mathbf{T}$ is an element of **LCA** one can see that G^* must remain inside **LCA** due to the properties of the exponential. Thus we can consider the double exponential of G relative to $\mathbf{T}$, $\mathbf{T}^{\mathbf{T}^G}$, which of course again is an element of **LCA**. Again for ease of notation we will write $\mathbf{T}^{\mathbf{T}^G}$ as G^{**}.

As mentioned in the previous section, easily, $(\hat{\hat{\ }})$ is a natural transformation, i.e. the following diagram commutes,

$$\begin{array}{ccc} G & \xrightarrow{(\hat{\hat{\ }})} & G^{**} \\ \downarrow f & & \downarrow f^{**} \\ G & \xrightarrow{(\hat{\hat{\ }})} & G^{**} \end{array}$$

But is $(\hat{\hat{\ }})$ a natural isomorphism inbetween G and G^{**}? We prove this by proving Pontrjagin Duality for full subcategories of **LCA** and then using methods from category theory and homological algebra to extend the concrete duality to larger categories and then eventually to **LCA**. Our path is described by the following diagram,

$$\begin{tikzcd}
 & & \mathbf{LCA} & & \\
 & & & & \text{Compactly Generated} \arrow[ull, "\text{exactness}"'] \\
\text{Discrete} \arrow[uurr, "\text{exactness}"] & & & & \text{Compact} \arrow[u, "\text{exactness}"'] \\
 & & \text{Elementary} \arrow[dll] \arrow[drr] \arrow[uurr, "\text{exactness}"] & & \\
\text{Discrete Elementary} \arrow[uu, "\varinjlim"] & & & & \text{Compact Elementary} \arrow[uu, "\varprojlim"'] \\
 & & \mathbf{T}, \mathbf{Z}, \mathbf{R}, \mathbf{Z}_n \arrow[uu, "\oplus"'] & &
\end{tikzcd}$$

6. Duality for $\mathbf{T}, \mathbf{Z}, \mathbf{R}, \mathbf{Z}_n$.

Thus we begin at the bottom of our diagram by showing that $\mathbf{T}$, $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{Z}_n$ obey Pontrjagin Duality. Thus note the following,

(1) $\mathbf{T} \cong \mathbf{Z}^*$ with $x \in \mathbf{T}$ corresponding to the mapping $n \mapsto nx$.
(2) $\mathbf{Z} \cong \mathbf{T}^*$ with $n \in \mathbf{Z}$ corresponding to the mapping $x \mapsto nx$ of $\mathbf{T}$.
(3) $\mathbf{R} \cong \mathbf{R}^*$ with $x \in \mathbf{R}$ corresponding to the mapping $y \mapsto xy + \mathbf{Z}$ of $\mathbf{R}$.
(4) $\mathbf{Z}_n \cong \mathbf{Z}_n^*$ with $x \in \mathbf{Z}_n$ corresponding to the mapping $y \mapsto xy$.

For proofs of all four of these isomorphisms see the appendix. Thus one can see that $G \cong G^{**}$ holds in a natural way for $G = \mathbf{T}, \mathbf{Z}, \mathbf{R}, \mathbf{Z}_n$.

7. Duality for Elementary Groups in $\mathbf{LCA}$.

Now we define the elementary groups.

Definition 7.1 (Elementary Groups)**.** *An elementary group $G \in \mathbf{LCA}$ is a group isomorphic to $\mathbf{T}^i \oplus \mathbf{Z}^h \oplus \mathbf{R}^k \oplus F$, where F is a finite abelian group. The full subcategory of elementary groups in* $\mathbf{LCA}$ *will be denoted by* $\mathbf{E}$.

To show duality for the elementary groups, all we need to do is lift the duality among the components of the direct sum to the entire direct sum. This can occur if and only if $(\hat{\ })$ (and thus $(\hat{\hat{\ }})$) preserves direct sums,

Lemma 7.1. $(\hat{\ })$ *preserves direct sums.*

Proof. See appendix. □

Theorem 7.2. $(\hat{\hat{\ }})$ *restricted to* $\mathbf{E}$ *is an isomorphism.*

Proof. Given $G \in \mathbf{E}$, due to the previous lemma G^{**} must still be a direct sum implying through easy manipulations that G^{**} obeys Pontrjagin Duality since all of it's constituant parts Pontrjagin Duality. □

8. Duality for Compact Groups in LCA.

We define a morphism $f \in \mathrm{Hom}_{\mathbf{LCA}}(G, H)$ to be **proper** if f is an open function.

Proposition 8.1. *If $f : G \to H$ is proper and $f(G)$ is open then f^* is proper.*

Proof. Given a compact neighborhood M of 0 in $G \in \mathbf{LCA}$ and W a closed neighborhood of 0 in $\mathbf{T}$ such that W contains no proper subgroups, define $K_{M,W}$ as the set of mappings,

$$K_{M,W} = \{\alpha \in G^* : \alpha(M) \subset W\}$$

Note that $K_{M,W}$ forms a basis of compact neighborhoods of 0 in G^*.

Let M be a compact neighborhood of 0 in $f(G)$. Since f is proper and G is locally compact, we can find a compact neighborhood N of 0 in G such that $f(N) = M$. Then we know that $K_{N,W}$ is a neighborhood of 0 in G^* and that $f^*(K_{M,W}) = K_{N,W} \cap f^*(H^*)$ is a compact neighborhood of 0 in $f^*(H^*)$. Thus we can conclude that f^* is proper by considering f^* upon the open subgroup of H^* generated by $K_{M,W}$ and then apply the open mapping theorem[1]. □

Proposition 8.2. $(\hat{\ })$ *takes a short proper exact sequence,*

$$0 \to K \xrightarrow{i} G \xrightarrow{j} H \to 0$$

to a sequence,

$$0 \leftarrow K^* \xleftarrow{i^*} G^* \xleftarrow{j^*} H^* \leftarrow 0$$

in which j^* is proper and exactness holds at G^* and H^*. If in addition K is an open subgroup of G, then the sequence induced by $(\hat{\ })$ is also proper exact.

Proof. First note that $\ker i^*$ must be $\varphi \in \mathrm{Hom}(G, T)$ which are trivial on K. Furthur since $i^* j^* = ji$, any $\mathrm{im}\, i = \ker j$, we can see that $i^* j^*(H^*)$ consists of mappings in G^* which are trivial on K. Thus $\mathrm{im}\, j^* = \ker i^*$.

Then remembering that surjections are epimorphisms in **LCA**, we know that j^* must be injective since given $\varphi_1, \varphi_2 \in \mathrm{Hom}(H, T)$,

$$\begin{aligned} j^*(\varphi_1) &= j^*(\varphi_2) \\ \varphi_1 j &= \varphi_2 j \\ \varphi_1 &= \varphi_2 \end{aligned}$$

Thus we have exactness at G^* and H^*. Now, because $\mathbf{T}$ is a divisible group[2], any $\varphi \in \mathrm{Hom}(H, \mathbf{T})$ extends to a not necessarily continuous homomorphism in $\mathrm{Hom}(G, \mathbf{T})$. If H is an open subgroup, then any such extension will be continuous on G, implying that i^* will be a surjection. □

Remembering the $K_{M,W}$ notation of the proof of Proposition 8.1, we see that if G is discrete and $M = \{0\}$ in G, then $K_{M,W} = G^*$, so G^* must be compact. On the other hand, if G is compact and $M = G$, then $K_{M,W} = \{0\}$, so G^* must be discrete.

[1] Theorem 5.29, p. 42 Hewitt and Ross [1963]

[2] i.e. an abelian group G such that for any $n \in \mathbf{N}$ and every $g \in G$, there exists $y \in G$ such that $ny = g$.

We now let A be the subcategory of discrete groups in **LCA** and A_0 be the subcategory of discrete elementary groups[3]. C is the subcategory of compact groups and C_0 the subcategory of compact elementary groups [4] in **LCA**.

Then we define a directed set, directed system, and direct limit,

Definition 8.1 (Directed Set)**.** *A* **directed set** *is a two tuple $(A, \leq)$ where A is a nonempty set and $\leq$ is a reflexive and transitive binary relation $\leq$ such that every $x, y \in A$ has an upper bound. Another name for $\leq$ is a pre-order.*

Definition 8.2 (Directed System)**.** *Let $(I, \leq)$ be a directed set. Define a* **direct system** *over I to be a two tuple (A, f_{ij}) where $A = \{A_i : i \in I\}$ and $f_{ij} : A_i \to A_j$ is a homomorphism for all $i \leq J$ such that f_{ii} is the identity on A_i and $f_{ik} = f_{jk} \circ f_{ij}, \forall i \leq j \leq k$.*

Definition 8.3 (Direct Limit)**.** *Let $S = (X_i, f_ij)$ be a direct system of objects and morphisms in $\mathcal{C}$. Then a direct limit of S is a 2-tuple (X, φ_i) where X is an object in $\mathcal{C}$ and φ_i consists of a family of morphisms $\varphi_i : X_i \to X$ satisfying $\varphi_i = \varphi_j \circ f_{ij}$.*

This property must be universal i.e. for any other pair $(Y, \phi_i,$ there must exist a unique morphism $u : X \to Y$ such that the following diagram commutes for all i, j,

$$\begin{tikzcd}
X_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & X_j \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\
& X \arrow[d, "u"] & \\
& Y &
\end{tikzcd}$$

Usually if the direct system (X_i, f_{ij}) is understood, the direct limit is denoted,

$$X = \lim_{\to} X_i$$

Note that a directed set I can be viewed as a category by declaring for $i, j \in I$ that $\mathrm{Hom}(i, j)$ consists of exactly one element if $i \leq j$ and is empty otherwise. In terms of our above definitions, we then can see that a direct system in A_0 is a covariant functor U from a directed set to A_0. We will write U_i for $U(i)$ and u_{ij} for $U(\mathrm{Hom}(i, j)) = \mathrm{Hom}(U_i, U_j)$.

Let DA_0 be the collection of direct systems in A_0 whose morphisms are all injective. DA_0 becomes a category by considering as its objects the covariant functor mappings $U : I \to A_0$ and as the morphisms, mappings from $U : I \to A_0$ to $V : J \to A_0$ where I, J are directed sets to be a pair (m, φ) where $m : I \to J$ is a functor in the category of directed sets[5] and φ is natural transformation from U to $V \circ m$.

Some well known properties of abelian groups include the fact that each element U of DA_0 has a direct limit $\lim\limits_{\to} U_i$ which will be an object of A. In fact, a necessary and sufficient condition for a discrete group G to be isomorphic to $\lim\limits_{\to} U_i$ is the existence of injective morphisms $g_i : U_i \to G$, one for each i, such that $g_j \circ u_{ij} = g_i$

[3] i.e. groups isomorphic to $\mathbf{Z}^j \oplus F$

[4] groups isomorphic to $\mathbf{T}^i \circ F$

[5] i.e. an order preserving map.

whenever $i \leq j$, and the union of the images $g_i(U_i)$ is all of G. The result is that $\varinjlim : DA_0 \to A$ can be regarded as a covariant functor, since if (m, φ) is a functor from U to V in DA_0, the universal property of $\varinjlim U_i$ guarantees existence of a unique morphism $\varinjlim (m, \varphi)\colon \varinjlim U_i \to \varinjlim V_i$ making the following diagram commute for every $i \in I$,

$$\begin{tikzcd} U_i \arrow[r] \arrow[d, "\varphi_i"'] & \varinjlim U_i \arrow[d, "\varinjlim (m,\varphi)"] \\ V_{m(i)} \arrow[r] & \varinjlim V_i \end{tikzcd}$$

Similarly, an inverse system in C_0 is a contravariant functor U from a directed set to C_0. The category of all inverse systems in C_0 all of whose morphisms are surjective is denoted by IC_0. A morphism from $U : I \to C_0$ to $V : J \to C_0$ in IC_0 is a pair (m, φ) where $m : J \to I$ is a functor and $\varphi : U \circ m \to V$ is a natural transformation.

Also, any inverse system in IC_0 has a projection limit $\varprojlim U_i$ which will be an object of C. Further, any object G of C is isomorphic to $\varprojlim U_i$ if and only if there exists a surjective morphism $g_i : G \to U_i$ for each i such that $u_{ij} \circ g_j = g_i$ whenever $i \leq j$ and the intersection of the kernels of the g_i is $\{0\}$ in G[6]. Again in this situation $\varprojlim (m, \varphi)$ for a morphism (m, φ) in IC_0 is defined by the universal property, and $\varprojlim$ becomes a covariant functor. We call DA_0 and IC_0 convergence structures on A and C, respectively.

By Proposition 8.2, if $U \in DA_0$ then $(\hat{\ }) \circ U \in IC_0$. This correspondence $U \mapsto (\hat{\ }) \circ U$ gives us a functor which we donate by $D(\hat{\ }) : DA_0 \to IC_0$. Similarly, $(\hat{\ })$ induces the functor $I(\hat{\ }) : IC_0 \to DA_0$.

Proposition 8.3. *$(\hat{\ })\varinjlim, \varprojlim \circ D(\hat{\ }) : DA_0 \to C$ are naturally isomorphic.*

Proof. Let $U \in DA_0$ and $G = \varinjlim U_i$ with $g_i : U_i \to G$ the associated injections. We must show that $G^* \cong \varprojlim (U_i^*)$. It is clear that each $g_i^* : G^* \to U_i^*$ is surjective. Let $0 \neq \alpha \in G^*$. We shall show the existence of an index i with $g_i^*(\alpha) \neq 0$. We know that $\alpha(x) \neq 0$ for some $x \in G$ and thus $x = g_i(y)$ for some i and $y \in G_i$. Then for this i,

$$g_i^*(\alpha)(y) = \alpha(g_i(y)) \neq 0.$$

So,

$$(\varinjlim U_i)^* \cong \varprojlim (U_i^*).$$

The fact that we have a natural isomorphism follows from the universal property. $\square$

Proposition 8.4. *$(\hat{\ }) \circ \varprojlim, \varinjlim \circ I(\hat{\ }) : IC_0 \to A$ are naturally isomorphic.*

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[6]the usual condition is that every neighborhood of 0 in G contain $\ker g_i$ for some i but if the intersection of the $\ker g_i$ is $\{0\}$ and N is any open neighborhood of 0 in G, then by compactness we can find a finite number of g_i whose kernels when intersected is contained in N. Picking an index j greater than these i gives us $\ker g_j \subset N$.

Proof. Let $U \in IC_0$ and $G = \varprojlim U_i$ with $g_i : G \to U_i$ the associated surjections. Clearly each g_i^* is injective. To show that G^* is isomorphic with $\varinjlim (U_i^*)$, we must show that every $\alpha \in G^*$ is equal to $g_i^*(\beta)$ for some i and some $\beta \in U_i^*$. Let W be a neighborhood of 0 in $\mathbf{T}$ containing no proper subgroups of $\mathbf{T}$. Let M be a neighborhood of 0 in G with $\alpha(M) \subset W$. Then we may find U_i with $\ker g_i \subset M$. Then $\alpha(\ker g_i) = 0$, so α factors over g_i for some $\beta \in U_i^*$. But $\beta \circ g_i = g_i^*(\beta)$ so we are done. □

Proposition 8.5. *The category C_0 is dense in C, i.e. there exists a functor $S : C \to IC_0$ such that the functor $\varprojlim \circ S$ and the identity functor on C are naturally isomorphic.*

Proof. This follows from the Peter-Weyl theorem, which says that for our abelian case that any $g \in \mathrm{Hom}(G, \mathbf{T})$ for G in C separate the points of G. Let $G \in C$ and define $S(G) \in IC_0$ to be the collection of quotient groups G/K of G which are in C_0. We order them by $G/K \leq G/N$ if $N \subset K$. Note that $G/(K \cap N)$ is isomorphic to a subgroup of $(G/K) \oplus (G/N)$, so we have a directed set. We can define $g_{KN} : G/N \to G/K$ to be the natural projection when $N \subset K$. For a morphism $f : G \to H$ in C, $S(f)$ is defined as follows: H/K in $S(H)$ corresponds to $G/f^{-1}(K)$ in $S(G)$, and the map $G/F^{-1}(K) \to H/K$ is the natural map induced by f. Then if $x \in G$, let $\alpha \in G^*$ with $\alpha(x) \neq 0$. Then G/K is in $S(G)$, where $K = \ker \alpha$, and $g_K(x) \neq 0$, where $g_K : G \to G/K$ is the canonical map. The collection g_K exhibits G as $\varprojlim S(G)$. □

Now we again consider our two covariant functors id and $(\hat{}\,\hat{})$.

Theorem 8.6. *Pontrjagin Duality holds in C.*

Proof. $(\hat{}\,\hat{})_{C_0}$ is already an isomorphism since C_0 consists of elementary groups. By Proposition 1.18 in Hofmann [1968], $(\hat{}\,\hat{})_{C_0}$ extends uniquely to a natural transformation between id_C and $(\hat{}\,\hat{})_C$. This extension must also be an isomorphism. But $(\hat{}\,\hat{})_C$ already extends $(\hat{}\,\hat{})_{C_0}$ so $(\hat{}\,\hat{})_C$ must be an isomorphism. □

9. Duality of Compactly Generated Groups in LCA.

Define **CG** as the full subcategory of compactly generated groups in **LCA**.

Lemma 9.1. *Suppose $G \in \mathbf{CG}$ is generated by the compact neighborhood M of 0 in G. Then there is a subgroup K of G, $K \cong Z^n$ for some n, such that $K \cap M = \{0\}$ and G/K is compact.*

Proof. This is Lemma 2.42 in Rudin [1962]. □

Proposition 9.2. *If $G \in \mathbf{CG}$ then $(\hat{}\,\hat{})_G$ is injective.*

Proof. Let $x \in G$ such that $x \neq 0$. Apply the lemma to $M \cup \{x\}$, which is also a compact neighborhood of 0 which generates G. The coset $x + K$ is not the identity element in the compact group G/K. Therefore, there is a mapping $\alpha \in \mathrm{Hom}(G/K, \mathbf{T})$ such that $\alpha(x + K) \neq 0$. Composing α with the canonical projection $G \to G/K$ gives us an element of $\mathrm{Hom}(G, \mathbf{T})$ which is not trivial on x. Therefore $(\hat{}\,\hat{})_G(x) \neq 0$. □

Theorem 9.3. *Pontrjagin duality holds in* **CG**.

Proof. Let $G \in \mathbf{CG}$. Let M be a compact neighborhood of 0 in G and let $S \cong Z^n$ be a subgroup of G such that $S \cap M = \{0\}$ and G/S is compact as gauranteed by the lemma above. Let $Q = G/S$ and $p : G \to Q$ the canonical map. Let N be a compact symmetric neighborhood of 0 in G such that $N + N + N \subset M$. Then p maps N homeomorphically onto $p(N)$. Since Q is compact and p is proper, there is a comapct subgroup Q_1 of Q such that $Q_1 \subset p(N)$ and Q/Q_1 is compact elementary by Proposition 8.5. Then $p^{-1}(Q_1)$ is a closed subgroup of G contained in $S + N$. Letting $K = p^{-1}(Q_1) \cap N$, we have by the choice of N that K is a compact subgroup of G satisfying $p(K) = Q_1$.

Let $H = G/K$. We shall show that H is an elementary group. First p gives rise to the proper surjection $H \to Q/Q_1$ with kernel $S + K$ which is discrete in H by the construction of K. Therefore H is locally isomorphic with Q/Q_1, which in turn is locally isomorphic with $\mathbf{R}^n$ for some n. This means we have an isomorphism $f : B \to V$ where B is an open ball about 0 in $\mathbf{R}^n$ and V is a neighborhood of 0 in H. Then we can extend f to a proper surjective homomorphism $g : \mathbf{R}^n \to H_1$, where H_1 is the open subgroup of H generated by V by defining $g(x) = nf(x/n)$ for $x \in \mathbf{R}^n$ and n large enough such that $x/n \in B$. Thus $H_1 \cong \mathbf{R}^a \oplus \mathbf{T}^b$ for some integers a and b (a quotient group of $\mathbf{R}^n$). Since H_1 is a divisible open subgroup of H, we can obtain a morphism $H \to H_1$ which is the identity on H_1 such that $H \cong H_1 \oplus (H/H_1)$. But H/H_1 is an elementary group since it is discrete and compactly generated. Therefore H is also elementary. Consider the following commutative diagram:

(1)
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K \arrow[r, "i"] \arrow[d, "(\hat{}\hat{})_K"] & G \arrow[r, "j"] \arrow[d, "(\hat{}\hat{})_G"] & H \arrow[r] \arrow[d, "(\hat{}\hat{})_H"] & 0 \arrow[d] \\
0 \arrow[r] & K^{**} \arrow[r, "i^{**}"] & G^{**} \arrow[r, "j^{**}"] & H^{**} \arrow[r] & 0
\end{tikzcd}$$

Now $K \in C$ and $H \in \mathbf{E}$ so $(\hat{}\hat{})_H$ and $(\hat{}\hat{})_K$ are isomorphisms, while $(\hat{}\hat{})_G$ is injective by Propositon 9.2. Since i is injective, we conclude that i^{**} is injective. Now briefly consider $i^* : G^* \to K^*$. If i^* were not surjective, then there would be a nontrivial mapping on $K^*/i^*(G^*)$. Composing with the canonical projection $K^* \to K^*/i^*(G^*)$, we get a mapping on K^* which is trivial on $i^*(G^*)$. This character would then be in the kernel of i^{**}, contradicting the injectivity of i^{**}. Therefore, i^* is surjective and thus Propositon 8.2 tells us that the induced sequence,

$$0 \leftarrow K^* \stackrel{i^*}{\leftarrow} G^* \stackrel{j^*}{\leftarrow} H^* \leftarrow 0$$

is a proper exact sequence since K^* is discrete. Thus H^* can be regarded as an open subgroup of G^*, and so the lower sequence in the diagram (1) is also proper exact. Thus the 5-lemma and the open mapping theorem show that $(\hat{}\hat{})_G$ is an isomorphism and we are done. □

10. Duality for Locally Compact Groups.

We begin with discrete groups.

Proposition 10.1. *There exists a functor $T : A \to DA_0$ such that $\varinjlim \circ T$ and the identity functor on A are naturally isomorphic.*

Proof. Every abelian group is the direct limit of its finitely generated subgroups. The functor T assigns to each G in A the direct system (U_i) of finitely generated subgroups of G ordered by the relation $i \leq j$ if $U_i \subset U_j$. A morphism $f : G \to H$ in A is carried by T to $T(f)$ which maps each finitely generated subgroup of G via f's restriction to the subgroups image in H. □

Theorem 10.2. *Pontrjagin Duality holds in A.*

Proof. This is Symmetric to Theorem 8.6. □

Theorem 10.3 (Pontrjagin Duality)**.** $(\hat{\ }\hat{\ })$ *is a natural equivalence.*

Proof. Let $G \in \mathbf{LCA}$. Let M be a compact neighborhood of 0 in G and let K be the subgroup of G generated by M. Let $H = G/K$. Since K is open, the induced sequence,

$$0 \leftarrow K^* \leftarrow G^* \leftarrow H^* \leftarrow 0$$

is proper exact via Proposition 8.2. Consider again the diagram (1) for this specific K, G, H. We have exactness at K^{**} and G^{**} in the bottom row. H is discrete since K is open, so $(\hat{\ }\hat{\ })_H$ is an isomophrism, and so j^{**} is surjective. Both rows of the diagram are proper exact and $(\hat{\ }\hat{\ })_K$ is also an isomorphism. Thus again using the 5-lemma, $(\hat{\ }\hat{\ })_G$ is an isomorphism. □

11. Appendix

Lemma 11.1. $(\hat{\ })$ *preserves direct sums.*

Proof. Let $G, H \in \mathbf{LCA}$ and let $G \oplus H$ be the ordinary group theoretic "direct sum" of G and H. Let i_G, i_H be the canonical embeddings of G and H into $G \oplus H$ and let p_G and p_H be the canonical projections of $G \oplus H$ onto G and H, respectively. $G \oplus H$ can be characterzied by the following commutative diagram,

$$\begin{tikzcd} & G \arrow[d, "i_G"] \arrow[dr, "1_G"] & \\ H \arrow[r, "i_H"] \arrow[dr, "1_H"'] & G \oplus H \arrow[r, "p_G"] \arrow[d, "p_H"] & G \\ & H & \end{tikzcd}$$

with the additional requirement that,

$$i_G \circ p_G \oplus i_H \circ p_H = 1_{G \oplus H}$$

Since $(\hat{\ })$ is an additive contravariant functor, applying $(\hat{\ })$ to the diagram results in the following,

$$\begin{tikzcd} & G^* & \\ H^* & G^* \oplus H^* \arrow[l, "i_H^*"'] \arrow[u, "i_G^*"] & G^* \arrow[l, "p_G^*"'] \arrow[ul, "1_G^*"'] \\ & H^* \arrow[u, "p_H^*"] \arrow[ul, "1_H^*"] & \end{tikzcd}$$

$(\hat{\ })$ will preserve the direct sum if and only if the diagram still commutes and the dual requirement is fullfilled. First we note that the commutativity of the diagram is preserved since commutativity is preserved in each of the triangles. This can be seen since without losing generality, given $\varphi \in H^*$,

$$\begin{aligned} i_H^*(p_H^*(\varphi)) &= i_H^*(\varphi p_H) \\ &= \varphi p_H i_H \\ &= \varphi \end{aligned}$$

Then we finish by noting that the dual requirement is fulfilled as follows, let $\varphi \in G^* \oplus H^*$

$$\begin{aligned} (p_G^*(i_G^*) \oplus p_H^*(i_H^*))(\varphi) &= 1_G^* \oplus 1_H^* \\ &= 1_{G^* \oplus H^*} \end{aligned}$$

□

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