MATHEMATICS 332: ALGEBRA — ASSIGNMENT 2

Reading: Gallian, chapters 0, 1, 2, 3, 10

Problems:

1. Let n be a positive integer. Is each of the following subsets of $\mathrm{GL}_n(\mathbb{R})$ a subgroup? When the answer is *no*, give some explanation. For one part, the answer depends on n.

(a) The set of symmetric matrices ($a_{ji} = a_{ij}$) in $\mathrm{GL}_n(\mathbb{R})$.
(b) The set of trace-zero matrices ($\sum_{i=1}^{n} a_{ii} = 0$) in $\mathrm{GL}_n(\mathbb{R})$.
(c) The set of upper-triangular matrices ($a_{ij} = 0$ if $i > j$) in $\mathrm{GL}_n(\mathbb{R})$.
(d) The set of diagonal matrices ($a_{ij} = 0$ if $i \neq j$) in $\mathrm{GL}_n(\mathbb{R})$.

2. Let $c \in \mathbb{R}_{>0}$ be a positive real number. Consider the set-with-operation $(G_c, \cdot)$ where the set is

$$G_c = \mathbb{Z} \times \mathbb{R} = \{(m, x) : m \in \mathbb{Z},\ x \in \mathbb{R}\},$$

and the operation is (omitting the "$\cdot$")

$$(m, x)(n, y) = (m + n, x + c^m y).$$

(a) Show that $(G_c, \cdot)$ is a group. (Don't bother explaining why the operation returns values in the same set.)
(b) For which (if any) values of c is the group commutative?
(c) For which (if any) values of c does the nonempty subset $\mathbb{Z} \times \mathbb{Q}$ of G_c form a subgroup under the operation?

3. Let G be a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$. Show that G is commutative.

4. For any pair of real numbers x and y, define the *double embedding* of the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as another column vector having three entries,

$$\iota : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad \iota \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}.$$

(a) Does ι surject? Does ι map to a basis of $\mathbb{R}^3$?
(b) Show that there exists a unique function

$$f : \mathrm{GL}_2(\mathbb{R}) \longrightarrow \mathrm{GL}_3(\mathbb{R})$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^2 & \xrightarrow{\ \cdot\ } & \mathbf{R}^2 \\ \Big\downarrow{\scriptstyle (f,\iota)} & & \Big\downarrow{\scriptstyle \iota} \\ \mathrm{GL}_3(\mathbb{R}) \times \mathbb{R}^3 & \xrightarrow{\ \cdot\ } & \mathbf{R}^3 \end{array}$$

That is, the desired relation is $f(m)\cdot(\iota v) = \iota(m\cdot v)$ for all $m \in \mathrm{GL}_2(\mathbb{R})$ and $v \in \mathbb{R}^2$. (If your solution does not cite part (a) then it can not possibly be complete. Also, don't forget to show that $f(m)$ lies in $\mathrm{GL}_3(\mathbb{R})$, i.e., that it is invertible.)

(c) Show that f is a homomorphism from $\mathrm{GL}_2(\mathbb{R})$ to $\mathrm{GL}_3(\mathbb{R})$. (Again, if your solution does not cite part (a) then it can not possibly be complete.)

5. (a) Let n be a positive integer. Let $h \in \mathrm{GL}_n(\mathbb{C})$ be hermitian, meaning that $h^* = h$ where h^* is the transpose-conjugate of h. Consider the set of matrices that preserve the inner product defined by h,

$$U(h) = \{m \in \mathrm{GL}_n(\mathbb{C}) : m^* h m = h\}.$$

Show that $U(h)$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. This subgroup is the *unitary group* of h.

(b) Show that the map

$$f : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{M}_n(\mathbb{C}), \quad m \longmapsto m^{-\top}$$

is an endomorphism of $\mathrm{GL}_n(\mathbb{C})$, and in fact an automorphism because it is its own inverse. (Such an automorphism is called an *involution*.)

(c) Show that the restriction of the map f from part (b) to $U(h)$ gives an isomorphism

$$f : U(h) \xrightarrow{\sim} U(\overline{h}^{-1}).$$