MULTILINEAR ALGEBRA: THE EXTERIOR PRODUCT

This writeup is drawn closely from chapter 28 of Paul Garrett's text **Abstract Algebra**, available from Chapman and Hall/CRC publishers and also available online at Paul Garrett's web site.

Throughout the writeup, let A be a commutative ring with 1. Every A-module is assumed to have the *unital* property that $1_A \cdot x = x$ for all x in the module.

1. Alternating and Skew-Symmetric Multilinear Maps

Definition 1.1. *Let M be an A-module. Let k be a positive integer and let*

$$M^{\times k} = M \times \cdots \times M$$

denote the k-fold product of M with itself, again an A-module. A multilinear map of A-modules,

$$\phi : M^{\times k} \longrightarrow X$$

is **alternating** *if it vanishes whenever any two if its arguments are equal,*

$$\phi(\cdots, m, \cdots, m, \cdots) = 0,$$

and is **skew-symmetric** *if its value is negated whenever any two its arguments are exchanged,*

$$\phi(\cdots, m', \cdots, m, \cdots) = -\phi(\cdots, m, \cdots, m', \cdots).$$

Proposition 1.2. *Consider a multilinear map of A-modules,*

$$\phi : M^{\times k} \longrightarrow X.$$

If ϕ is alternating map then it is skew-symmetric. If ϕ is skew-symmetric and the only $x \in X$ such that $x + x = 0_X$ is $x = 0_X$ then ϕ is alternating.

Proof. Since only two variables are involved at a time, we may take $k = 2$. Compute, using only multilinearity, that for all $m, m' \in M$,

$$\phi(m + m', m + m') = \phi(m, m) + \phi(m, m') + \phi(m', m) + \phi(m', m').$$

If ϕ is alternating then the relation becomes

$$0 = \phi(m, m') + \phi(m', m),$$

showing that ϕ is skew-symmetric. If ϕ is skew-symmetric then in particular for all $m \in M$,

$$\phi(m, m) = -\phi(m, m).$$

Thus $\phi(m, m) + \phi(m, m) = 0$ and so $\phi(m, m) = 0$ by our hypothesis on X, showing that ϕ is alternating. $\square$

2. The Exterior Product: Mapping Property and Uniqueness

Definition 2.1 (Mapping Property of the Exterior Product)**.** *Let M be an A-module and let k be a positive integer. The* k**th exterior product of** M **over** A *is another A-module and an alternating multilinear map to it,*

$$\varepsilon : M^{\times k} \longrightarrow \bigwedge\nolimits_A^k M,$$

having the following property: For every alternating A-multilinear map from the cartesian product to an A-module,

$$\phi : M^{\times k} \longrightarrow X,$$

*there exists a unique A-*linear *map from the exterior product to the same module,*

$$\Phi : \bigwedge\nolimits_A^k M \longrightarrow X,$$

such that $\Phi \circ \varepsilon = \phi$, i.e., such that the following diagram commutes,

$$\begin{tikzcd} \bigwedge_A^k M \arrow[dr, dashed, "\Phi"] & \\ M^{\times k} \arrow[u, "\varepsilon"] \arrow[r, "\phi"] & X. \end{tikzcd}$$

Note: In contrast to the fact that various A-modules can be tensored together, the kth exterior product involves k copies of *one* A-module.

Proposition 2.2 (Uniqueness of the Exterior Product)**.** *Let M be an A-module. Given two kth exterior products of M over A,*

$$\varepsilon_1 : M^{\times k} \longrightarrow E_1 \quad \text{and} \quad \varepsilon_2 : M^{\times k} \longrightarrow E_2,$$

there is a unique A-module isomorphism $i : E_1 \longrightarrow E_2$ such that $i \circ \varepsilon_1 = \varepsilon_2$, i.e., such that the following diagram commutes,

$$\begin{tikzcd} & M^{\times k} \arrow[dl, "\varepsilon_1"'] \arrow[dr, "\varepsilon_2"] & \\ E_1 \arrow[rr, dashed, "i"] & & E_2. \end{tikzcd}$$

The proof is virtually identical to several proofs that we have seen before. Indeed, one can encode a single meta-argument to encompass all of them.

3. The Exterior Product: Existence

Proposition 3.1 (Existence of the Exterior Product)**.** *Let M be an A-module and let k be a positive integer. Then a kth exterior product $\varepsilon : M^{\times k} \longrightarrow \bigwedge_A^k M$ exists.*

Proof. Let $\tau : M^{\times k} \longrightarrow \bigotimes_A^k M$ be the kth tensor product of M with itself. Let S be the A-submodule of $\bigotimes_A^k M$ generated by all monomials

$$\cdots \otimes m \otimes \cdots \otimes m \otimes \cdots$$

in which an element appears more than once. Consider the quotient $Q = (\bigotimes_A^k M)/S$ and the quotient map

$$q : \bigotimes\nolimits_A^k M \longrightarrow Q.$$

The claim is that

$$q \circ \tau : M^{\times k} \longrightarrow Q \quad \text{is a } k\text{th exterior product.}$$

To prove the claim, we must verify the desired mapping property. Thus, consider any alternating multilinear map of A-modules,

$$\phi : M^{\times k} \longrightarrow X.$$

The mapping property of the tensor product $\bigotimes_A^k M$ gives a unique commutative diagram in which the map Ψ is A-linear,

$$\begin{tikzcd} \bigotimes_A^k M \arrow[dr, dashed, "\Psi"] & \\ M^{\times k} \arrow[u, "\tau"] \arrow[r, "\phi"] & X. \end{tikzcd}$$

This diagram does not show that $\tau : M^{\times k} \longrightarrow \bigotimes_A^k M$ is the exterior product because τ is not alternating. However, compute that since the diagram commutes and ϕ is alternating,

$$\Psi(\cdots \otimes m \otimes \cdots \otimes m \otimes \cdots,) = \phi(\cdots, m, \cdots, m, \cdots) = 0.$$

Thus Ψ factors through the quotient Q,

$$\begin{tikzcd} \bigotimes_A^k M \arrow[r, "q"] \arrow[dr, "\Psi"'] & Q \arrow[d, dashed, "\Phi"] \\ & X. \end{tikzcd}$$

Concatenate the previous two diagrams and then consolidate to get the desired diagram,

$$\begin{tikzcd} & Q \arrow[d, "\Phi"] \\ M^{\times k} \arrow[ur, "q\circ\tau"] \arrow[r, "\phi"] & X. \end{tikzcd}$$

Furthermore, if also $\widetilde{\Phi} \circ q \circ \tau = \phi$ then $\widetilde{\Phi} \circ q = \Phi \circ q$ by the uniqueness property of the tensor product, and thus $\widetilde{\Phi} = \Phi$ since q surjects.

Finally, $q \circ \tau$ is alternating by the definition of S (exercise). And so the data $\bigwedge_A^k M = Q$ and $\varepsilon = q \circ \tau$ satisfy the exterior product mapping property. $\square$

4. Tangible Descriptions

For any $(m_1, \cdots, m_k) \in M^{\times k}$, the image $\varepsilon(m_1, \cdots, m_k) \in \bigwedge_A^k M$ is denoted

$$m_1 \wedge \cdots \wedge m_k.$$

Some relations in $\bigwedge_A^k M$ are

$$\begin{aligned} &m_1 \wedge \cdots \wedge (m_i + m_i') \wedge \cdots \wedge m_k \\ &\qquad = (m_1 \wedge \cdots \wedge m_i \wedge \cdots \wedge m_k) + (m_1 \wedge \cdots \wedge m_i' \wedge \cdots \wedge m_k), \\ &m_1 \wedge \cdots \wedge am_i \wedge \cdots \wedge m_k = a(m_1 \wedge \cdots \wedge m_i \wedge \cdots \wedge m_k), \\ &m_1 \cdots \wedge m_i \wedge \cdots \wedge m_i \wedge \cdots \wedge m_k = 0, \\ &m_1 \cdots \wedge m_j \wedge \cdots \wedge m_i \wedge \cdots \wedge m_k = -(m_1 \cdots \wedge m_i \wedge \cdots \wedge m_j \wedge \cdots \wedge m_k). \end{aligned}$$

As an application of the mapping property, we prove

Proposition 4.1 (Exterior Product Generators)**.** *Let M be an A-module. Then the exterior product $\varepsilon : M^{\times k} \longrightarrow \bigwedge_A^k M$ is generated by the monomials $m_1 \wedge \cdots \wedge m_k$ where each $m_i \in M$. Furthermore, if a set of generators of M over A is $\{e_i\}$ (where the index set I is well ordered) then a set of generators of $\bigwedge_A^k M$ is*

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}.$$

Proof. Let $E = \bigwedge_A^k M$, let S be the A-submodule of E generated by the monomials, let $Q = E/S$ be the quotient, and let $q : E \longrightarrow Q$ be the quotient map. Also, let $z : M^{\times k} \longrightarrow Q$ and $Z : E \longrightarrow Q$ be the zero maps. Certainly

$$Z \circ \varepsilon = z,$$

but also, since $\varepsilon(M^{\times k}) \subset S$,

$$q \circ \varepsilon = z.$$

Thus the uniqueness statement in the mapping property of the exterior product gives $q = Z$. In other words, S is all of E.

As for the second statement in the proposition, the first statement shows that any monomial in $\bigwedge_A^k M$ takes the form of the left side of the equality

$$\left(\sum_{i_1} a_{i_1} e_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_k} a_{i_k} e_{i_k}\right) = \sum_{i_1, \cdots, i_k} a_{i_1} \cdots a_{i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

That is, the equality shows that any monomial in $\bigwedge_A^k M$ is a linear combination of $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$. By skew symmetry, we may assume that $i_1 < \cdots < i_k$. Since any element of $\bigwedge_A^k M$ is a linear combination of monomials in turn, we are done. $\square$

Lemma 4.2. *Let M be an A-module, not necessarily finitely-generated or free. Let k and m be positive integers, and let $n = k + m$. Then there is a unique bilinear map*

$$\beta : \bigwedge\nolimits_A^k M \times \bigwedge\nolimits_A^m M \longrightarrow \bigwedge\nolimits_A^n M$$

whose action on pairs of monomials is

$$\beta(m_1 \wedge \cdots \wedge m_k, m_{k+1} \wedge \cdots \wedge m_n) = m_1 \wedge \cdots \wedge m_k \wedge m_{k+1} \wedge \cdots \wedge m_n.$$

Proof. For any $m_{k+1}, \cdots, m_n$, the map

$$M^{\times k} \longrightarrow \bigwedge\nolimits_A^n M, \quad (m_1, \cdots, m_k) \longmapsto m_1 \wedge \cdots \wedge m_k \wedge m_{k+1} \wedge \cdots \wedge m_n$$

is multilinear and alternating, and so it gives rise to a linear map

$$\beta_1 : \bigwedge\nolimits_A^k M \longrightarrow \bigwedge\nolimits_A^n M, \quad m_1 \wedge \cdots \wedge m_k \longmapsto m_1 \wedge \cdots \wedge m_k \wedge m_{k+1} \wedge \cdots \wedge m_n.$$

Similarly, for any $m_1, \cdots, m_k$, there is a linear map

$$\beta_2 : \bigwedge\nolimits_A^m M \longrightarrow \bigwedge\nolimits_A^n M, \quad m_{k+1} \wedge \cdots \wedge m_n \longmapsto m_1 \wedge \cdots \wedge m_k \wedge m_{k+1} \wedge \cdots \wedge m_n.$$

Thus the map $\beta = \beta_1 \times \beta_2$ is bilinear as desired. $\square$

Proposition 4.3 (Exterior Product Rank)**.** *Let M be a free A-module of rank n. Let k be a positive integer. Then $\bigwedge_A^k M$ is a free A-module of rank $\binom{n}{k}$.*

Proof. Let $\{e_i : i = 1, \cdots, n\}$ be an A-basis of M, and let $\{\widehat{e}_j : j = 1, \cdots, n\}$ be the dual basis,

$$\widehat{e}_j : M \longrightarrow A, \quad \widehat{e}_j(e_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

First take $k = n$. The previous proposition shows that

$$\bigwedge\nolimits_A^n M = A \cdot e_1 \wedge \cdots \wedge e_n,$$

but the proposition does not show that the right side is free. We need to show that if $a \cdot e_1 \wedge \cdots \wedge e_n = 0$ then $a = 0$. Construct a map from $M^{\times n}$ to A that antisymmetrizes over all permutations $\pi \in S_n$,

$$\phi : M^{\times n} \longrightarrow A, \quad \phi(m_1, \cdots, m_n) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \widehat{e}_{\pi(1)}(m_1) \cdots \widehat{e}_{\pi(n)}(m_n).$$

Then clearly $\phi(e_1, \cdots, e_n) = 1$. Also, formal verifications confirm that ϕ is multilinear and alternating. Thus the mapping property of the nth exterior product gives a commutative diagram

$$\begin{tikzcd} \bigwedge_A^n M \arrow[dr, dashed, "\Phi"] & \\ M^{\times n} \arrow[u, "\varepsilon"] \arrow[r, "\phi"'] & A. \end{tikzcd}$$

Now if $a \cdot e_1 \wedge \cdots \wedge e_n = 0$ in $\bigwedge_A^n M$ then a itself must be 0:

$$a = a\phi(e_1, \cdots, e_n) = a\Phi(e_1 \wedge \cdots \wedge e_n) = \Phi(a \cdot e_1 \wedge \cdots \wedge e_n) = \Phi(0) = 0.$$

This shows that $\bigwedge_A^n M$ is free of rank 1 as desired.

If $1 \leq k < n$ then assume a linear dependence

$$\sum_I a_I e_I = 0 \quad \text{in } \bigwedge\nolimits_A^k M,$$

where the sum is over k-tuples $I = (i_1, \cdots, i_n)$ with $1 \leq i_1 < \cdots < i_k \leq n$ and a typical summand is

$$a_I e_I = a_{(i_1, \cdots, i_k)} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

For a given multi-index J let $\overline{e_J} \in \bigwedge_A^{n-k} M$ denote the exterior product of e_1 through e_n but with e_J removed,

$$\overline{e_J} = e_1 \wedge \cdots \wedge \overline{e_{j_1}} \wedge \cdots \wedge \overline{e_{j_k}} \wedge \cdots \wedge e_n.$$

Then, using the bilinear map β from the lemma,

$$\begin{aligned} 0_{\wedge^n M} &= \beta(0_{\wedge^k M}, \overline{e_J}) = \beta(\sum_I a_I e_I, \overline{e_J}) = \sum_I a_I \beta(e_I, \overline{e_J}) = \sum_I a_I e_I \wedge \overline{e_J} \\ &= \pm a_J e_1 \wedge \cdots \wedge e_n, \end{aligned}$$

so that $a_J = 0_A$ as argued a moment ago. □

5. Exterior Powers of a Map

Consider an A-linear map,

$$f : M \longrightarrow M'.$$

The map

$$\varepsilon' \circ f^{\times k} : M^{\times k} \longrightarrow \bigwedge\nolimits_A^k M'$$

is readily seen to be multilinear and alternating. The mapping property of $\bigwedge_A^k M$ thus gives a unique linear map,

$$\begin{tikzcd} \bigwedge_A^k M \arrow[r, dashed, "f^{\wedge k}"] & \bigwedge_A^k M' \\ M^{\times k} \arrow[u, "\varepsilon"] \arrow[r, "f^{\times k}"] & (M')^{\times k}. \arrow[u, "\varepsilon'"'] \end{tikzcd}$$

In symbols, the formula for $f^{\wedge k}$ is

$$f^{\wedge k}(m_1 \wedge \cdots \wedge m_k) = f(m_1) \wedge \cdots \wedge f(m_k).$$

6. The Determinant Revisited

Let n be a positive integer, and let M be free of rank n over A. Thus $\bigwedge_A^n M$ is free of rank 1 over A, and so for any A-linear map $f : M \longrightarrow M$, the nth exterior power

$$f^{\wedge n} : \bigwedge\nolimits_A^n M \longrightarrow \bigwedge\nolimits_A^n M, \quad f^{\wedge n}(m_1 \wedge \cdots \wedge m_n) = f(m_1) \wedge \cdots \wedge f(m_n)$$

is multiplication by some ring element. This element is the **determinant** of f, written $\det f$. Thus

$$f(m_1) \wedge \cdots \wedge f(m_n) = \det f \cdot (m_1 \wedge \cdots \wedge m_n) \quad \text{for all } m_1, \cdots, m_n,$$

where $\det f \in A$.

Take a basis $\beta = (e_1, \cdots, e_n)$ of M. For any A-linear map $f : M \longrightarrow M$, let the matrix of f with respect to β have columns $m_1, \cdots, m_n$. Write $\det(m_1, \cdots, m_n)$ for $\det f$. Then since $m_i = f(e_i)$ for $i = 1, \cdots, n$, the previous display gives

$$m_1 \wedge \cdots \wedge m_n = \det(m_1, \cdots, m_n)\, e_1 \wedge \cdots \wedge e_n.$$

The n-fold exterior product of module elements $e_1 \wedge \cdots \wedge e_n$ on the right side of the equality is fixed. Because the product $m_1 \wedge \cdots \wedge m_n$ on the left side is multilinear and alternating as a function of $m_1, \cdots, m_n$ and equals $e_1 \wedge \cdots \wedge e_n$ when $(m_1, \cdots, m_n) = (e_1, \cdots, e_n)$, the scalar $\det(m_1, \cdots, m_n)$ on the right side is also multilinear and alternating as a function of $m_1, \cdots, m_n$ and equals 1 when $(m_1, \cdots, m_n) = (e_1, \cdots, e_n)$. That is, $\det f$ is multilinear, alternating, and normalized as a function of the columns of any matrix of f, which is to say that $\det f$ is indeed the familiar determinant from linear algebra.

Let $f, g : M \longrightarrow M$ be A-linear, so that the composition $fg : M \longrightarrow M$ is A-linear as well, and compute that for any $m_1, \cdots, m_n \in M$,

$$\begin{aligned} \det(fg) \cdot m_1 \wedge \cdots \wedge m_n &= fg(m_1) \wedge \cdots \wedge fg(m_n) \\ &= \det f \cdot g(m_1) \wedge \cdots \wedge g(m_n) \\ &= \det f \det g \cdot m_1 \wedge \cdots \wedge m_n. \end{aligned}$$

This shows that

$$\det fg = \det f \det g \quad \text{for all linear } f, g : M \longrightarrow M.$$

7. Application of the Determinant: Cramer's Rule

Let A be a commutative ring with 1. Let n be a positive integer. Consider an n-by-n matrix and two column vectors, all having entries in A,

$$m = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

The goal is to solve the equation $mx = b$ for x, given m and b.

The matrix-by-vector product mx is a linear combination of the columns of m, weighted by the entries of x,

$$mx = \sum_j c_j x_j \quad \text{where } c_j \text{ is the } j\text{th column of } m.$$

Thus if $mx = b$ then we have for each $i \in \{1, \cdots, n\}$, since the determinant is multilinear and alternating,

$$\begin{aligned} &\det(c_1, \cdots, c_{i-1}, b, c_{i+1}, \cdots, c_n) \\ &\quad = \det(c_1, \cdots, c_{i-1}, \sum_j c_j x_j, c_{i+1}, \cdots, c_n) \\ &\quad = \sum_j x_j \det(c_1, \cdots, c_{i-1}, c_j, c_{i+1}, \cdots, c_n) \\ &\quad = x_i \det(m). \end{aligned}$$

Thus if $\det(m) \in A^\times$ then the solution x of the equation $mx = b$ is uniquely determined,

$$x_i = \det(c_1, \cdots, c_{i-1}, b, c_{i+1}, \cdots, c_n) \det(m)^{-1}, \quad i = 1, \cdots, n.$$

8. The Classical Adjoint Revisited

In linear algebra, the so-called "classical adjoint" of a square matrix is another square matrix whose (i, j)th entry is $(-1)^{i+j}$ times the determinant of the original matrix with its jth row and its ith column deleted. The circumstance that this bewildering construction is called an *adjoint* despite seeming unrelated to the usual definition of the adjoint ($\langle m^* v, v' \rangle = \langle v, mv' \rangle$ for all v, v'), and the fact that the classical adjoint nearly inverts the original matrix, are typically taken for granted by the student. This section explains how the name and the properties of the classical adjoint are perfectly lucid in the context of multilinear algebra.

Let n be a positive integer, and let M be free of rank n over A. Recall the bilinear pairing

$$M \times \bigwedge\nolimits_A^{n-1} M \longrightarrow \bigwedge\nolimits_A^n M$$

given by

$$\langle m_1, m_2 \wedge \cdots \wedge m_n \rangle = m_1 \wedge m_2 \wedge \cdots \wedge m_n.$$

Any A-linear map $f : M \longrightarrow M$ has an **adjugate** A-linear map $f^{\mathrm{adg}} : M \longrightarrow M$ defined as the adjoint of $f^{\wedge(n-1)}$ under the bilinear pairing. That is, the defining property of the adjugate is that for all $m_1, \cdots, m_n \in M$,

$$f^{\mathrm{adg}}(m_1) \wedge m_2 \wedge \cdots \wedge m_n = m_1 \wedge f^{\wedge(n-1)}(m_2 \wedge \cdots \wedge m_n).$$

Compute that for any A-linear map $f : M \longrightarrow M$ and for all $m_1, \cdots, m_n \in M$,

$$\begin{aligned}(f^{\mathrm{adg}} f)(m_1) \wedge m_2 \wedge \cdots \wedge m_n &= f(m_1) \wedge f^{\wedge(n-1)}(m_2 \wedge \cdots \wedge m_n) \\ &= f^{\wedge n}(m_1 \wedge m_2 \wedge \cdots \wedge m_n) \\ &= \det f\,(m_1 \wedge m_2 \wedge \cdots \wedge m_n) \\ &= (\det f \cdot m_1) \wedge m_2 \wedge \cdots \wedge m_n.\end{aligned}$$

That is,

$$f^{\mathrm{adg}} f : M \longrightarrow M \quad \text{is multiplication by } \det f.$$

Now suppose that the ring A is an integral domain, so that it has a quotient field k. Suppose also that $\det f \neq 0$ in A. Extend the scalars of M from A to k by forming the tensor product

$$M' = k \otimes_A M.$$

Then multiplication by $\det f$ is is invertible as an endomorphism of M', and thus the relation $f^{\mathrm{adg}} f = \det f \cdot \mathrm{id}_{M'}$ shows that so is f, giving

$$f^{\mathrm{adg}} = (\det f \cdot \mathrm{id}_{M'}) f^{-1}.$$

This equality shows that f^{adg} and f commute as endomorphisms of M', and so they commute as endomorphisms of M. Summarizing,

- $f^{\mathrm{adg}} : M \longrightarrow M$ is defined as the adjoint of $f^{\wedge(n-1)}$ under the bilinear pairing of M and $\bigwedge_A^{n-1} M$.
- $f^{\mathrm{adg}} f = \det f \cdot \mathrm{id}_M$.
- If A is an integral domain and $\det f \neq 0$ then f^{adg} commutes with f.

9. A Uniqueness Result Revisited

Let A be a PID. Consider a finitely generated free A-module and a submodule,

$$\begin{aligned}F &= Ae_1 \oplus \cdots \oplus Ae_m \oplus \cdots \oplus Ae_n, \\ S &= \mathfrak{a}_1 e_1 \oplus \cdots \oplus \mathfrak{a}_m e_m,\end{aligned}$$

where

$$\mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_m.$$

(Of course it is tacit that the ideals are nonzero.) With exterior products available, we can give an intrinsic structural description of the ideals $\mathfrak{a}_1, \cdots, \mathfrak{a}_m$ in terms of F and S. Thus F and S uniquely determine the ideals.

Already $\mathfrak{a}_1$ has been described intrinsically. It is the image of S under a functional $F \longrightarrow A$, and it contains the image of S under every functional $F \longrightarrow A$. Next consider the exterior products

$$\begin{aligned}F^{\wedge 2} &= A(e_1 \wedge e_2) \oplus \cdots \oplus A(e_{n-1} \wedge e_n), \\ S^{\wedge 2} &= \mathfrak{a}_1\mathfrak{a}_2(e_1 \wedge e_2) \oplus \cdots \oplus \mathfrak{a}_{m-1}\mathfrak{a}_m(e_{m-1} \wedge e_m).\end{aligned}$$

Certainly $\mathfrak{a}_1\mathfrak{a}_2$ is the image of $S^{\wedge 2}$ under a functional $F^{\wedge 2} \longrightarrow A$, and because it contains all ideals $\mathfrak{a}_i\mathfrak{a}_j$, it contains the image of $S^{\wedge 2}$ under every functional $F^{\wedge 2} \longrightarrow A$. Thus $\mathfrak{a}_1\mathfrak{a}_2$ is intrinsic to F and S, and consequently the second elementary divisor $\mathfrak{a}_2$ is intrinsic to F and S as well, being the annihilator of $\mathfrak{a}_1/\mathfrak{a}_1\mathfrak{a}_2$. The argument for the remaining elementary divisors is more of the same, going up to $F^{\wedge m} = Ae_1 \wedge \cdots \wedge e_m \oplus \cdots$ and $S^{\wedge m} = \mathfrak{a}_1 \cdots \mathfrak{a}_m e_1 \wedge \cdots \wedge e_m$. Note that m itself is described intrinsically as the highest exponent of a nonzero exterior power of S.

Now we can see that the ingredients of our earlier uniqueness proof—alternating multilinear maps, in particular determinants of square subblocks that never referenced the part of the larger module F indexed by basis elements beyond the dimension of S—are natural in the multilinear algebra environment, and we understand that their role in the earlier proof was to create just-enough exterior product structure to obtain the desired result.

10. Example: Filling Out a Matrix

Let our basic data be a PID A and positive integers k and n with $k < n$. Suppose that we are given are given a k-by-n matrix with entries in A,

$$[a_{ij}]_{(1,\cdots,k)\times(1,\cdots,n)}.$$

Let the determinants of the corresponding k-by-k minors of the matrix be

$$d_J = \det([a_{ij}]_{(1,\cdots,k)\times J}), \quad J = (j_1, \cdots, j_k),\ j_1 \leq \cdots \leq j_k.$$

Assume that the determinants are altogether coprime,

$$\gcd(\{d_J\}) = 1.$$

The problem is to add $n-k$ rows to the matrix and obtain a resulting n-by-n matrix having determinant 1. If the given k rows were simply the standard basis vectors e_1 through e_k then the problem would be trivial. We will see that the structure theorem for finitely generated modules over a PID provides a coordinate system in which the problem is indeed trivial as just described.

To begin the solution, write the k rows of the given matrix as vectors, letting $(e_1, \cdots, e_n)$ denote the standard basis of $A^{\oplus n}$ as usual,

$$f_i = \sum_{j=1}^{n} a_{ij} e_j, \quad i = 1, \cdots, k.$$

Since $\gcd(\{d_J\}) = 1$, the vectors are linearly independent. View $A^{\oplus n}$ as a rank-n free A-module F and consider the rank-k submodule S spanned by the given vectors, i.e., by the rows of the given matrix,

$$F = \bigoplus_{j=1}^{n} Ae_j, \qquad S = \bigoplus_{i=1}^{k} Af_i.$$

The kth exterior powers $F^{\wedge k}$ and $S^{\wedge k}$ are, using the notations $e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}$, $I = (1, \cdots, k)$, and $f_I = f_1 \wedge \cdots \wedge f_k$,

$$F^{\wedge k} = \bigoplus_J Ae_J, \qquad S^{\wedge k} = Af_I.$$

Recall the minor-determinants d_J. We claim that in fact

$$f_I = \sum_J d_J e_J.$$

(For example, when $k = 2$ and $n = 3$ this is the formula for the "cross product" from vector calculus.) Indeed, each term $c_J e_J$ of the product $f_I = f_1 \wedge \cdots \wedge f_k$ is the product $f_{1,J} \wedge \cdots \wedge f_{k,J}$ where each $f_{i,J} = \sum_{j \in J} a_{i,j} e_j$ is the J-projection of f_i. The coefficient of the latter product is a multilinear, alternating, and normalized function of the rows of $[a_{ij}]_{(1,\cdots,k)\times J}$, and so it is the determinant d_J. Thus the previous two displays combine to say

$$F^{\wedge k} = \bigoplus_J A e_J, \qquad S^{\wedge k} = A \cdot \sum_J d_J e_J.$$

Next, the most recent display and the given condition $\gcd(\{d_J\}) = 1$ combine to show that the quotient $F^{\wedge k}/S^{\wedge k}$ is torsion-free: the crux of the argument is that for any $a, b, \{c_J\}$ in A with $a \neq 0$,

$$\begin{aligned} a \sum c_J e_J = b \sum d_J e_J &\implies a \mid a c_J = b d_J \text{ for all } J \\ &\implies a \mid b \qquad\qquad\qquad \text{since } \gcd(\{d_J\}) = 1 \\ &\implies b = a\beta \text{ for some } \beta \\ &\implies \sum c_J e_J = \beta \sum d_J e_J \quad \text{by cancellation.} \end{aligned}$$

Now return to the original A-modules F and S. The structure theorem for finitely generated modules over a PID provides (as constructively as the gcd algorithm in A is constructive) a basis $(g_1, \cdots, g_n)$ of F and nonzero ideals $\mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_k$ of A such that

$$F = \bigoplus_{j=1}^{n} A g_j, \qquad S = \bigoplus_{i=1}^{k} \mathfrak{a}_i g_i.$$

Consequently, using the notations $g_J = g_{j_1} \wedge \cdots \wedge g_{j_k}$ and $g_I = g_1 \wedge \cdots \wedge g_k$,

$$F^{\wedge k} = \bigoplus_J A g_J, \qquad S^{\wedge k} = \mathfrak{a}_1 \cdots \mathfrak{a}_k g_I,$$

so that

$$F^{\wedge k}/S^{\wedge k} = (A/\mathfrak{a}_1 \cdots \mathfrak{a}_k) g_I \oplus \bigoplus_{J \neq I} A g_J.$$

Since this quotient is torsion free, in fact $\mathfrak{a}_1 = \cdots = \mathfrak{a}_k = A$. That is,

$$F = \bigoplus_{j=1}^{n} A g_j, \qquad S = \bigoplus_{j=1}^{k} A g_j.$$

But also $S = \bigoplus_{j=1}^{k} A f_j$ from before. Thus $(f_1, \cdots, f_k, g_{k+1}, \cdots, g_n)$ is a basis of $A^{\oplus n}$. After scaling g_n by an element of $A^\times$ if necessary, the n-by-n matrix with these basis elements as its rows has determinant 1, and the problem is solved.