STEREOGRAPHIC PROJECTION IS CONFORMAL

Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

be the unit sphere, and let $\mathbf{n}$ denote the north pole $(0, 0, 1)$. Identify the complex plane $\mathbb{C}$ with the (x, y)-plane in $\mathbb{R}^3$. The *stereographic projection* map,

$$\pi : S^2 - \mathbf{n} \longrightarrow \mathbb{C},$$

is described as follows: place a light source at the north pole $\mathbf{n}$. For any point $p \in S^2 - \mathbf{n}$, consider a light ray emanating downward from $\mathbf{n}$ to pass through the sphere at p. The ray also meets the plane, and the point where it hits is $\pi(p)$. That is,

$$\pi(p) = \ell(\mathbf{n}, p) \cap \mathbb{R}^2,$$

where $\ell(\mathbf{n}, p) = \{(1-t)\mathbf{n} + tp : t \in \mathbb{R}\}$ is the line through $\mathbf{n}$ and p. (See figure 1.)

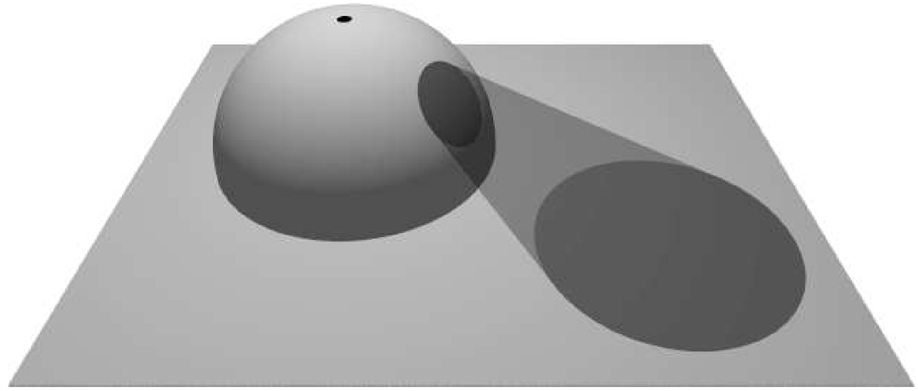

Figure 1. Stereographic projection

The formula for stereographic projection is

$$\pi(x, y, z) = \frac{x + iy}{1 - z}.$$

Indeed, the point $(1-t)\mathbf{n} + t(x, y, z)$ has last coordinate $1 - t + tz$. This equals 0 for $t = 1/(1-z)$, making the other coordinates $x/(1-z)$ and $y/(1-z)$, and the formula follows.

For the inverse map, take a point $q = (x, y, 0)$ in the plane. Since $\mathbf{n}$ and q are orthogonal, any point $p = (1-t)\mathbf{n} + tq$ on the line $\ell(\mathbf{n}, q)$ satisfies $|p|^2 = (1-t)^2 + t^2|q|^2$. This equals 1 for $t = 2/(|q|^2 + 1)$ (ignoring $t = 0$, which gives the north pole), showing that

$$\pi^{-1}(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

Stereographic projection is conformal, meaning that it preserves angles between curves. To see this, take a point $p \in S^2 \setminus \{\mathbf{n}\}$, let T_p denote the tangent plane to S^2 at p, and let $T_{\mathbf{n}}$ denote the tangent plane to S^2 at $\mathbf{n}$. Working first in the $\mathbf{0n}p$-plane (see figure 2), we have equal angles α and right angles between the radii and the tangent planes, hence equal angles β, hence equal angles β', and hence equal lengths b.

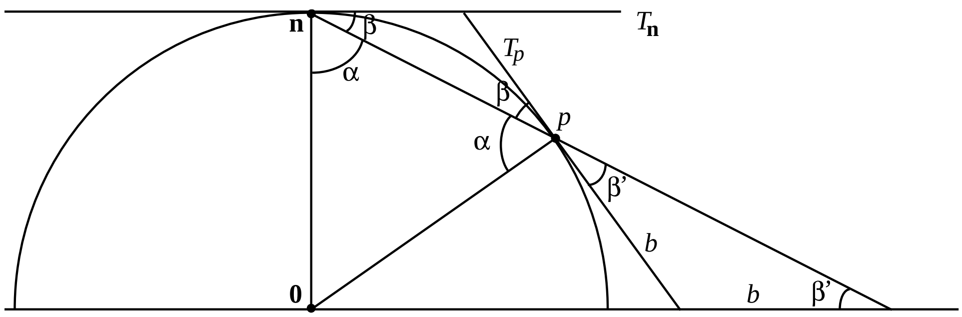

Figure 2. Side view of stereographic projection

Now let γ be a smooth curve on S^2 through p, let t be its tangent at p, and let $\hat{t}$ be the intersection of the plane containing $\mathbf{n}$ and t with $\mathbb{R}^2$. (See figure 3.) In fact $\hat{t}$ is the tangent to $\pi \circ \gamma$ at $\pi(p)$. To see this, note that π is the restriction of a rational, hence differentiable, map (also called π) from an $\mathbb{R}^3$-neighborhood of p to $\mathbb{R}^2$ that takes t to $\hat{t}$ near p. (A *neighborhood* of a point is an open set containing the point.) Since γ and t are curves in $\mathbb{R}^3$ with the same tangent t at p, it follows that $\pi \circ \gamma$ and $\pi \circ t = \hat{t}$ are curves in $\mathbb{R}^2$ with the same tangent at $\pi(p)$. Since $\hat{t}$ is its own tangent at $\pi(p)$, it is also the tangent to $\pi \circ \gamma$ there. The lengths b are equal, hence so are the angles θ, by right triangles. Repeating this analysis for a second curve $\tilde{\gamma}$ through p completes the proof.

For a continuation of this argument, showing that stereographic projection takes circles to circles, see *Geometry and the Imagination* by Hilbert and Cohn-Vossen.

Followup Exercises

- Illustrate the proof that stereographic projection is conformal when p lies in the lower hemisphere.
- The proof that stereographic projection is conformal tacitly assumed that t and $\hat{t}$ meet. Must they? What happens to the proof if they don't?
- Show that stereographic projection takes circles to circles.

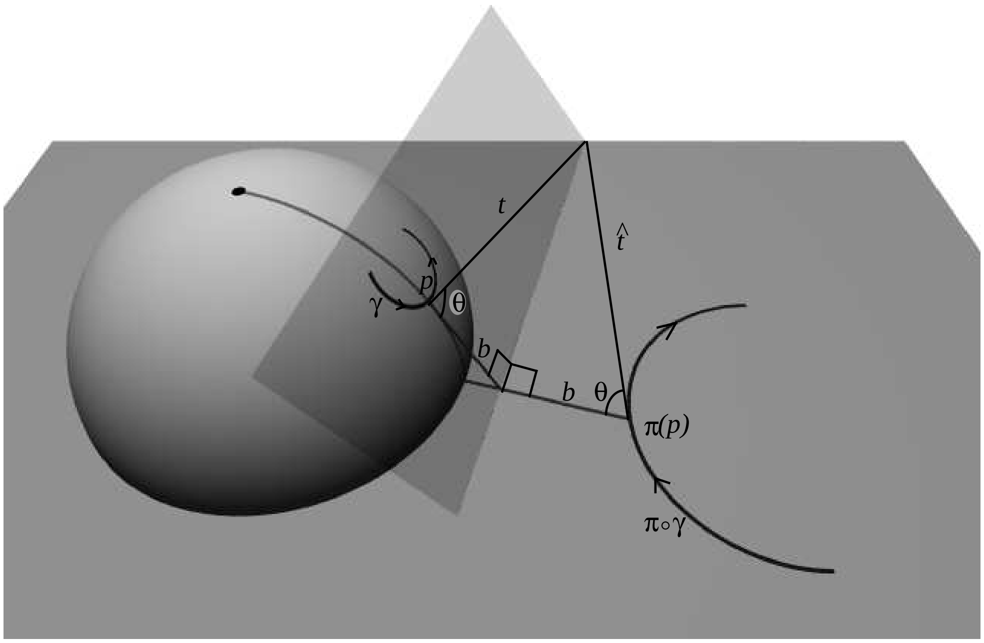

Figure 3. Stereographic projection is conformal