THE RATIO TEST

Consider a complex power series all of whose coefficients are nonzero,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n, \quad a_n \neq 0 \text{ for each } n.$$

Suppose that the limit

$$R = R(f) = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

exists in the extended nonnegative real number system $[0, \infty]$. We show that R is the radius of convergence of f,

> *$f(z)$ converges absolutely on the open disk of radius R about c, and this convergence is uniform on compacta, but $f(z)$ diverges if $|z-c| > R$.*

Not every power series has coefficients that are all nonzero, and even if all the coefficients are nonzero then the limit R needn't exist, so the statement here is only a partial result. For the full story, see this course's related writeup on the radius of convergence formula, involving an idea called the *limit superior.*

We freely take $c = 0$, and we proceed by cases.

1. The Case $0 \leq R < \infty$

If $0 < R < \infty$, let z vary through a compact subset K of the open disk of radius R about 0; this open disk is empty for $R = 0$. Thus, for some $\varepsilon > 0$,

$$|z| \leq R - 2\varepsilon, \quad z \in K.$$

Because $\lim_n |a_n|/|a_{n+1}| = R$, there is a starting index N such that

$$\begin{aligned} R - \varepsilon &< |a_N|/|a_{N+1}| \\ R - \varepsilon &< |a_{N+1}|/|a_{N+2}| \\ R - \varepsilon &< |a_{N+2}|/|a_{N+3}|, \end{aligned}$$

and so on. It follows that

$$\begin{aligned} |a_{N+1}| &< |a_N|/(R-\varepsilon) \\ |a_{N+2}| &< |a_{N+1}|/(R-\varepsilon) < |a_N|/(R-\varepsilon)^2 \\ |a_{N+3}| &< |a_{N+2}|/(R-\varepsilon) < |a_N|/(R-\varepsilon)^3, \end{aligned}$$

and in general

$$|a_{N+m}| < |a_N|/(R-\varepsilon)^m, \quad m = 1, 2, 3, \ldots,$$

from which

$$|a_{N+m} z^{N+m}| < |a_N|(R-2\varepsilon)^N \frac{(R-2\epsilon)^m}{(R-\varepsilon)^m}, \quad m = 1, 2, 3, \ldots.$$

Introduce the quantities $C = |a_N|(R-2\varepsilon)^N$ and $\rho = \frac{R-2\varepsilon}{R-\varepsilon} < 1$, and now the previous display is

$$|a_{N+m}z^{N+m}| < C\rho^m, \quad m = 1, 2, 3, \ldots .$$

The head of the sum of the absolute values of the terms of the power series satisfies the estimate

$$\sum_{n=0}^{N-1} |a_n z^n| \leq \sum_{n=0}^{N-1} |a_n|(R-2\epsilon)^n,$$

and the tail satisfies

$$\sum_{n=N}^{\infty} |a_n z^n| < C \sum_{m=0}^{\infty} \rho^m = \frac{C}{1-\rho}\,.$$

So $\sum_{n\geq 0} |a_n z^n|$ converges altogether. The convergence uniform over K because for $M \geq N$,

$$\sum_{n=M}^{\infty} |a_n z^n| < C\rho^{M-N} \sum_{m=0}^{\infty} \rho^m = \frac{C}{1-\rho}\rho^{M-N},$$

and as M goes to ∞, this goes to 0 independently of where z lies in K.

Now with $0 \leq R < \infty$, suppose that $|z| > R$. Because $\lim_n |a_n|/|a_{n+1}| = R$, there is a starting index N such that

$$|a_{N+m}|/|a_{N+m+1}| < |z|, \quad m = 0, 1, 2, \ldots,$$

and so, similarly to above,

$$|a_{N+m}| > |a_N|/|z|^m, \quad m = 1, 2, 3, \ldots,$$

from which, with $C = |a_N||z|^N > 0$,

$$|a_{N+m}z^{N+m}| > C, \quad m = 1, 2, 3, \ldots .$$

Thus $\sum_{n=0}^{\infty} a_n z^n$ diverges because its terms don't go to 0.

2. The Case $R = \infty$

Let z vary through any compact subset K of $\mathbb{C}$. Thus for some $B \geq 0$,

$$|z| \leq B, \quad z \in K.$$

Because $\lim_n |a_n|/|a_{n+1}| = \infty$, there is a starting index N such that

$$B + 1 < |a_{N+m}|/|a_{N+m+1}|, \quad m = 0, 1, 2, \ldots .$$

As above, but now with $C = |a_N|B^N$ and $\rho = \frac{B}{B+1}$,

$$|a_{N+m}z^{N+m}| < C\rho^m, \quad m = 1, 2, 3, \ldots .$$

From here the convergence argument is exactly as before. No divergence argument is needed here because the convergence holds everywhere.