TERMWISE DERIVATIVES OF POWER SERIES, SANS INTEGRALS

A direct argument, making no reference to integral representation, shows that any complex power series is termwise differentiable in its disk of convergence.

Consider a power series, centered at 0 without loss of generality, and consider also its termwise derivative,

$$p(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad q(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Assume that p has a positive radius of convergence, and let D denote its open disk of convergence, the open disk of convergence of q as well. Let z be any fixed point of D, so that all points ζ close enough to z also lie in D. We want to show that

$$\lim_{\zeta \to z} \frac{p(\zeta) - p(z)}{\zeta - z} = q(z).$$

There exists some positive r less than the radius of convergence of p such that $|z| < r$. Suppose also that $|\zeta| < r$, a condition that holds for all ζ close enough to z. For any nonnegative integer N, define the N-th partial sum and the N-th error of p,

$$s_N(p; z) = \sum_{n=0}^{N} a_n z^n, \qquad e_N(p; z) = \sum_{n=N+1}^{\infty} a_n z^n,$$

and define $s_N(q)$ and $e_N(q)$ similarly. Thus $p = s_N(p) + e_N(p)$ and $q = s_N(q) + e_N(q)$. We want a difference to go to 0 as ζ goes to z. Decompose it into three pieces,

$$\frac{p(\zeta) - p(z)}{\zeta - z} - q(z) = A_N(z, \zeta) + B_N(z, \zeta) + C_N(z),$$

where

$$\begin{aligned} A_N(z, \zeta) &= \frac{s_N(p; \zeta) - s_N(p; z)}{\zeta - z} - s_N(q; z), \\ B_N(z, \zeta) &= \frac{e_N(p; \zeta) - e_N(p; z)}{\zeta - z}, \\ C_N(z) &= -e_N(q; z). \end{aligned}$$

Let $\varepsilon > 0$ be given. For any fixed N, $A_N(z, \zeta)$ goes to 0 as ζ goes to z, because the polynomial $s_N(q)$ is the derivative of the polynomial $s_N(p)$. Also, because $\zeta^n - z^n = (\zeta - z) \sum_{j=0}^{n-1} \zeta^j z^{n-1-j}$, recalling that $|z| < r$ and $|\zeta| < r$,

$$|B_N(z, \zeta)| = \left| \sum_{n=N+1}^{\infty} a_n \frac{\zeta^n - z^n}{\zeta - z} \right| = \left| \sum_{n=N+1}^{\infty} a_n \sum_{j=0}^{n-1} \zeta^j z^{n-1-j} \right| \le \sum_{n=N+1}^{\infty} n |a_n| r^{n-1}.$$

The last sum in the previous display is the tail of a convergent series because $q(r)$ converges absolutely, and so $|B_N(z, \zeta)| < \varepsilon$ if the fixed N is large enough and ζ is close enough to z. Further, for large enough N, we have $|C_N(z)| < \epsilon$ because the power series $q(z)$ converges. Altogether, choosing N large enough establishes a statement that makes no reference to N:

$$\text{Given } \varepsilon > 0, \quad \left| \frac{p(\zeta) - p(z)}{\zeta - z} - q(z) \right| < 3\varepsilon \text{ for all } \zeta \text{ close enough to } z.$$

This is the desired result.

A variant argument is more direct, although it can appear more cluttered initially because it uses the difference-of-powers formula twice. We have

$$\frac{p(\zeta) - p(z)}{\zeta - z} - q(z) = \sum_{n=1}^{\infty} a_n \left(\frac{\zeta^n - z^n}{\zeta - z} - nz^{n-1} \right).$$

For $n = 1$, the term in parentheses is 0. For $n \geq 2$, it is

$$\begin{aligned} \frac{\zeta^n - z^n}{\zeta - z} - nz^{n-1} &= \left(\sum_{j=0}^{n-1} \zeta^{n-1-j} z^j \right) - nz^{n-1} = \sum_{j=0}^{n-2} (\zeta^{n-1-j} z^j - z^{n-1}) \\ &= \sum_{j=0}^{n-2} z^j (\zeta^{n-1-j} - z^{n-1-j}) = (\zeta - z) \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} \zeta^k z^{n-2-k}. \end{aligned}$$

Note how using the difference-of-powers formula twice has moved $\zeta - z$ from the denominator to the numerator. Again with $|z| < r$ and $|\zeta| < r$ where r is less than the radius of convergence, gather and estimate the terms,

$$\left| \frac{p(\zeta) - p(z)}{\zeta - z} - q(z) \right| \leq |\zeta - z| \sum_{n=2}^{\infty} |a_n| n^2 r^{n-2}.$$

The coefficient-size in this last series is $|a_n|n^2$ rather than $|a_n|$, but this growth has no effect on the radius of convergence. Thus the series on the right side of the inequality converges, and so the left side goes to 0 as ζ goes to z. That is, $p'(z)$ exists and equals $q(z)$. This is the desired result.