A CONVERGENCE CRITERION FOR INFINITE PRODUCTS

Let $\{z_n\}$ be a complex sequence, with $|z_n| < 1$ for all n. We show:

$$\boxed{\text{If } \sum_{n=1}^{\infty} |z_n| \text{ converges then } \prod_{n=1}^{\infty} (1 + z_n) \text{ converges and can be rearranged.}}$$

Because $\sum_{n=1}^{\infty} |z_n|$ converges, all but finitely many z_n satisfy $|z_n| \leq 1/2$. We freely work only with these z_n, for which

$$|\log(1 + z_n)| = \left|z_n(1 - z_n/2 + z_n^2/3 + \cdots)\right| \leq 2|z_n|.$$

Thus the sequence $\left\{\sum_{n=1}^{N} \log(1 + z_n)\right\}$ of partial sums of $\sum_{n=1}^{\infty} \log(1 + z_n)$ converges absolutely, and so it converges and can be rearranged. Consequently, because the complex exponential function is continuous, convergence and rearrangability also hold for the sequence

$$\left\{\exp^{\sum_{n=1}^{N} \log(1+z_n)}\right\} = \left\{\prod_{n=1}^{N} \exp^{\log(1+z_n)}\right\} = \left\{\prod_{n=1}^{N} (1 + z_n)\right\}.$$

This is the sequence of partial products of $\prod_{n=1}^{\infty}(1+z_n)$, and so the boxed statement is established.