SUM OF LOGARITHMS, LOGARITHM OF PRODUCT

We begin with an easy observation. Let $z_1, z_2, \dots$ lie in $\mathbb{C}$. Suppose that the sequence of their partial sums converges,

$$\{\sum_{n=1}^{N} z_n\} = \{s_N\} \overset{N}{\to} s.$$

Because the exponential is a continuous homomorphism from $(\mathbb{C}, +)$ to $(\mathbb{C}^\times, \cdot)$, the sequence of partial products of the exponentials converges correspondingly,

$$\{\prod_{n=1}^{N} e^{z_n}\} = \{p_N\} \overset{N}{\to} p \quad \text{where } p = e^s.$$

Briefly, if $\sum_n z_n = s$ then $\prod_n e^{z_n} = e^s$.

We want a converse result. Let $z_1, z_2, \dots$ lie in $\mathbb{C}^\times$. Suppose that the sequence of their partial products converges in $\mathbb{C}^\times$ (convergence to 0 is not convergence in the multiplicative group $\mathbb{C}^\times$),

$$\{\prod_{n=1}^{N} z_n\} = \{p_N\} \overset{N}{\to} p \neq 0.$$

We will show that for any appropriately sensible chosen branch of the complex logarithm, the sequence of partial sums of the logarithms converges to some logarithm of the product,

$$\{\sum_{n=1}^{N} \log z_n\} = \{s_N\} \overset{N}{\to} s, \quad \text{where } s \in \log p + 2\pi i\mathbb{Z}.$$

But because no branch of the logarithm is a homomorphism, no claim can be made that $s = \log p$ for the branch of log that is being used. Briefly, if $\prod z_n = p \neq 0$ then $\sum \log z_n \in \log p + 2\pi i\mathbb{Z}$.

To begin the argument, recall that we have a sequence $z_1, z_2, \dots$ in $\mathbb{C}^\times$ such that the sequence $\{\prod_{n=1}^{N} z_n\} = \{p_N\}$ of partial products converges to some p in $\mathbb{C}^\times$. Note that the multiplicative analogue of the nth term test applies to the sequence of partial products: Because $\{p_N\}$ converges to p in $\mathbb{C}^\times$, the individual terms z_N must converge to 1,

$$\{z_N\} = \left\{\frac{p_N}{p_{N-1}}\right\} \overset{N}{\to} \frac{p}{p} = 1.$$

Also, the disk D about p of radius $|p|/2$ misses 0, and all but finitely many of the partial products p_N lie in D, and the disk D' about 1 of radius $1/2$ misses 0, and all but finitely many of the z_N lie in D'. Take a branch cut that misses D and misses D' and misses the finitely many terms z_N and partial products p_N outside of the two disks. Specifically, the branch cut can be chosen as a ray from the origin, not $\mathbb{R}_{\geq 0}$, and then the logarithm can be chosen so that $\log 1 = 0$.

Because the sequence $\{p_N\}$ converges to p in the domain of our branch of the logarithm, which is continuous, also

$$\{\log p_N\} \stackrel{N}{\to} \log p.$$

But our concern is the convergence of the sum $\sum_n \log z_n$, whose partial sums $\sum_{n=1}^N \log z_n$ need not equal $\log p_N$, and so we need to consider how they differ. Because $\{\log p_N\}$ converges, we have

$$\{\log p_N - \log p_{N-1}\} \stackrel{N}{\to} 0.$$

Also, note that

$$\begin{aligned}\log p_N - \log p_{N-1} &= \log(p_{N-1}z_N) - \log p_{N-1}\\ &= \log p_{N-1} + \log z_N + 2\pi i k_N - \log p_{N-1} \quad \text{for some } k_N \in \mathbb{Z}\\ &= \log z_N + 2\pi i k_N.\end{aligned}$$

So now we have

$$\{\log z_N + 2\pi i k_N\} \stackrel{N}{\to} 0.$$

But $\{z_N\} \stackrel{N}{\to} 1$ and $\log 1 = 0$, so in fact

$$\{k_N\} \stackrel{N}{\to} 0.$$

Because each k_N is an integer, there exists a starting index N_o such that

$$k_N = 0 \quad \text{for all } N \geq N_o.$$

This shows that the logarithm does behave homomorphically on the partial products of large index,

$$\log p_N = \log p_{N-1} + \log z_N \quad \text{for all } N \geq N_o.$$

And one integer k captures the non-homomorphic behavior up to the starting index where the homomorphic behavior begins,

$$\sum_{n=1}^{N_o} \log z_n = \log p_{N_o} + 2\pi i k \quad \text{for some } k \in \mathbb{Z}.$$

Thus the sequence of partial sums of the logarithms of the sequence terms is, from the N_oth term onwards,

$$\{\sum_{n=1}^{N} \log z_n\}_{N \geq N_o} = \{\log p_N\}_{N \geq N_o} + 2\pi i k.$$

We know that the sequence $\{\log p_N\}$ converges to $\log p$, and so the claim is established,

$$\{\sum_{n=1}^{N} \log z_n\} \stackrel{N}{\to} \log p + 2\pi i k.$$