SIMPLE PROOF OF THE PRIME NUMBER THEOREM

This writeup is drawn from a writeup by Paul Garrett for his complex analysis course,

`http://www-users.math.umn.edu/~garrett/m/complex/notes_2014-15/09_prime_number_theorem.pdf`

Especially, the bibliography of the source writeup contains relevant papers of Chebyshev, Erdős, Garrett, Hadamard, Newman, de la Vallée Poussin, Riemann, Selberg, and Wiener.

The *prime-counting function*, a function of a real variable, is

$$\pi(x) = |\{p : p \leq x\}|.$$

That is, $\pi(x)$ equals the number of prime numbers that are at most x. The **Prime Number Theorem** states that

$$\boxed{\pi(x) \sim \frac{x}{\log(x)}}$$

meaning that $\lim_{x\to\infty} \pi(x)/(x/\log(x)) = 1$.

The *Chebyshev theta function*, also a function of a real variable, is

$$\vartheta(x) = \sum_{p \leq x} \log p.$$

A quick argument shows that $\vartheta(x) = \mathcal{O}(x)$, meaning that $\vartheta(x) \leq cx$ for some c and all large x; in fact, the argument produces such a c and the inequality holds for all x. A basic lemma of asymptotics specializes to show that if $\vartheta(x) \sim x$, meaning that $\lim_{x\to\infty} \vartheta(x)/x = 1$, then $\pi(x) \sim x/\log x$, giving the Prime Number Theorem. Thus the main work of this writeup is to go from $\vartheta(x) = \mathcal{O}(x)$ to $\vartheta(x) \sim x$. With $\zeta(s)$ the Euler–Riemann zeta function, the dominant term of $\zeta'(s)/\zeta(s)$ near $s = 1$ is a Dirichlet-like series closely related to $\vartheta(x)$. This fact combines with the convergence theorem in section 4 below to finish the proof.

Contents

1. Weak theta asymptotic

With $\vartheta(x) = \sum_{p \le x} \log p$ as above, a quick argument shows that

$$\boxed{\vartheta(x) = \mathcal{O}(x)}$$

as follows. For any positive integer n,

$$\prod_{n < p \le 2n} p \le \binom{2n}{n} < \sum_{j=0}^{2n} \binom{2n}{j} = 2^{2n},$$

and so

$$\vartheta(2n) - \vartheta(n) = \sum_{n < p \le 2n} \log p = \log \left(\prod_{n < p \le 2n} p \right) < 2n \log 2.$$

It follows that

$$\vartheta(2^m) < 2^{m+1} \log 2, \quad m \in \mathbb{Z}_{\ge 1},$$

and now, because any $x > 1$ satisfies $2^{m-1} < x \le 2^m$ for some such m,

$$\vartheta(x) < 2^{m+1} \log 2 < 4x \log 2.$$

So indeed $\vartheta(x) = \mathcal{O}(x)$.

2. Lemma on asymptotics, beginning of the proof

2.1. **Lemma.** The following lemma is elementary and ubiquitous in asymptotics.

Lemma 2.1. *Suppose that a sequence $\{c_n\}$ satisfies*

$$\sum_{n \le x} c_n \log n \sim rx \quad \textit{for some } r.$$

Then

$$\sum_{n \le x} c_n \sim \frac{rx}{\log x}\,.$$

Proof. Name the two sums in the lemma,

$$\theta(x) = \sum_{n \le x} c_n \log n \qquad \text{and} \qquad \varphi(x) = \sum_{n \le x} c_n.$$

Thus $\theta(x) \sim rx$, and we want to show that $\varphi(x) \sim rx/\log x$. Because the step function $\theta(x)$ jumps by $c_n \log n$ at each n, and the step function $\varphi(x)$ jumps by c_n at each n, we have for $t > 1$ in the sense of Stieltjes integration,

$$\mathrm{d}\varphi(t) = \frac{\mathrm{d}\theta(t)}{\log t}\,.$$

With "$*$" denoting a fixed, large enough lower limit of integration, and with a Stieltjes integral and integration by parts,

$$\varphi(x) \sim \int_{t=*}^{x} \mathrm{d}\varphi(t) = \int_{t=*}^{x} \frac{\mathrm{d}\theta(t)}{\log t} = \frac{\theta(t)}{\log t}\bigg|_{t=*}^{x} + \int_{t=*}^{x} \frac{\theta(t)}{t\log^2 t}\,\mathrm{d}t. \tag{1}$$

The boundary term is asymptotically $rx/\log x$, as desired for $\varphi(x)$, so what needs to be shown is that the last integral in (1) is $o(x/\log x)$.

Because $\theta(t)/t \sim r$ for large t, estimate the integral of $1/\log^2 t$, first breaking it into two pieces,

$$\int_{t=*}^{x} \frac{1}{\log^2 t}\,\mathrm{d}t = \int_{t=*}^{\sqrt{x}} \frac{1}{\log^2 t}\,\mathrm{d}t + \int_{t=\sqrt{x}}^{x} \frac{1}{\log^2 t}\,\mathrm{d}t.$$

For the first piece,

$$\int_{t=*}^{\sqrt{x}} \frac{1}{\log^2 t}\,\mathrm{d}t \le \sqrt{x}\int_{t=*}^{\sqrt{x}} \frac{1}{t\log^2 t}\,\mathrm{d}t = -\sqrt{x}\frac{1}{\log t}\bigg|_{t=*}^{\sqrt{x}} \sim \sqrt{x},$$

while for the second,

$$\int_{t=\sqrt{x}}^{x} \frac{1}{\log^2 t}\,\mathrm{d}t \le \frac{1}{\log^2 \sqrt{x}}(x-\sqrt{x}) \sim \frac{2x}{\log^2 x}.$$

Altogether $\int_{t=*}^{x} \mathrm{d}t/\log^2 t$ is $o(x/\log x)$. Because $\theta(t)/t = \mathcal{O}(1)$, the last integral in (1) is therefore $o(x/\log x)$ as well, and the argument is complete. □

2.2. **Beginning of the proof.** Consider the prime-indicator sequence, $\{c_n\} = \{c_1, c_2, \dots\}$ where

$$c_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

The Chebyshev theta function and the prime-counting function function are naturally re-expressed using this sequence,

$$\vartheta(x) = \sum_{n\le x} c_n \log n \qquad \text{and} \qquad \pi(x) = \sum_{n \le x} c_n.$$

Consequently the lemma reduces the Prime Number Theorem to showing that

$$\boxed{\vartheta(x) \sim x}$$

Already $\vartheta(x) = \mathcal{O}(x)$ is established, so the work is to go from this to the boxed result.

3. Euler–Riemann zeta function

3.1. **Zeta as a sum.** The Euler–Riemann zeta function is initially defined as a sum on an open right half plane,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

This sum converges absolutely on $\mathrm{Re}(s) > 1$ because $|n^{-s}| = n^{-\mathrm{Re}(s)}$, and hence it indeed converges on $\mathrm{Re}(s) > 1$. Each truncation $\sum_{n=1}^{N} n^{-s}$ of the sum is entire.

Let K denote a compact subset of $\mathrm{Re}(s) > 1$. There exists some $\sigma > 1$ such that $\mathrm{Re}(s) \geq \sigma$ on K, and so

$$\left|\sum_{n=N}^{\infty} n^{-s}\right| \leq \sum_{n=N}^{\infty} n^{-\sigma}, \quad s \in K.$$

This shows that the sum $\zeta(s)$ converges uniformly on K. Altogether, $\zeta(s)$ is holomorphic on $\mathrm{Re}(s) > 1$.

3.2. **Zeta as a product.** The Euler–Riemann zeta function has a second expression as a product of so-called Euler factors over the prime numbers,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1.$$

The equality of the product and sum expressions of $\zeta(s)$ for $\mathrm{Re}(s) > 1$ is a matter of the geometric series formula and the Fundamental Theorem of Arithmetic, as follows. Consider any positive integer k, let $p_1, \ldots, p_k$ denote the first k primes, compute

$$\begin{aligned}\prod_{i=1}^{k}(1 - p_i^{-s})^{-1} &= \prod_{i=1}^{k} \lim_{M_i \to \infty} \sum_{m_i=0}^{M_i} p_i^{-m_i s} = \lim_{M_1,\ldots,M_k \to \infty} \prod_{i=1}^{k} \sum_{m_i=0}^{M_i} p_i^{-m_i s} \\ &= \lim_{M_1,\ldots,M_k \to \infty} \sum_{\substack{n=\prod_{i=1}^k p_i^{m_i} \\ m_i \leq M_i \text{ each } i}} n^{-s} = \sum_{n=\prod_{i=1}^k p_i^{m_i}} n^{-s},\end{aligned}$$

and take the limit as k goes to ∞ to get the result, $\prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$. Now the product form of $\zeta(s)$ inherits the holomorphy of the sum form.

Also we can show that the product is a holomorphic function on $\mathrm{Re}(s) > 1$ with no reference to its matching the sum. Recall a general result for a product $\prod_{n=1}^{\infty}(1 + \varphi_n(s))$ with each φ_n holomorphic on a domain Ω, as follows.

Suppose that:
For every compact K in Ω
there is a summable sequence $\{x_n\} = \{x_n(K)\}$ in $\mathbb{R}_{\geq 0}$ such that
$|\varphi_n(s)| \leq x_n$ for all n, uniformly over $s \in K$.
Then $\prod_{n=1}^{\infty}(1 + \varphi_n(s))$ is holomorphic on Ω.

In our case, Ω is $\mathrm{Re}(s) > 1$, and $\varphi_n(s)$ is $(1 - p^{-s})^{-1} - 1 = (1 - p^{-s})^{-1} p^{-s}$ if n is a prime p, while $\varphi_n = 0$ if n is composite. Let K be a compact subset of $\mathrm{Re}(s) > 1$. There exists some $\sigma > 1$ such that $\mathrm{Re}(s) \geq \sigma$ on K. Let $\{x_n\} = \{2n^{-\sigma}\}$. For any prime p, for all $s \in K$,

$$|\varphi_p(s)| = |(1 - p^{-s})^{-1} p^{-s}| \leq 2p^{-\sigma} = x_p,$$

and $|\varphi_n(s)| = 0 \leq x_n$ for composite n and $s \in K$. Thus the product $\prod_p (1 - p^{-s})^{-1}$ is holomorphic on $\mathrm{Re}(s) > 1$, as claimed.

3.3. **Euler's proof.** Using the product form of $\zeta(s)$, consider the logarithm of the zeta function for s approaching 1 from the right,

$$\log \zeta(s) = \sum_p \log((1 - p^{-s})^{-1}) = \sum_p \sum_{m \geq 1} \frac{1}{mp^{ms}}. \tag{2}$$

This decomposes into two terms,

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + \sum_p \sum_{m\geq 2} \frac{1}{mp^{ms}}\,.$$

The sum form of $\zeta(s)$ shows that ζ diverges at 1, and hence so does $\log \zeta$ although more slowly. The second sum is bounded by 1,

$$\sum_p \sum_{m\geq 2} \frac{1}{mp^{ms}} < \sum_p \frac{1}{p^{2s}(1-p^{-s})} = \sum_p \frac{1}{p^s(p^s-1)} < \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1.$$

So the first sum $\sum_p p^{-s}$ is asymptotic to $\log \zeta(s)$ as s goes to 1, and consequently the prime numbers are dense enough to make the sum diverge at $s = 1$. This is a stronger result than the existence of infinitely many primes. For the Prime Number Theorem, we will similarly study $(\log \zeta(s))' = \zeta'(s)/\zeta(s)$ at $s = 1$.

3.4. **Continuation of zeta and its logarithmic derivative.** The function $\zeta(s)$ continues meromorphically to $\mathrm{Re}(s) > 0$, the only singularity of the extension being a simple pole at $s = 1$ with residue $\mathrm{res}_1\zeta(s) = 1$. The argument requires some estimation but isn't deep, as follows. For $\mathrm{Re}(s) > 1$, introduce the function

$$\psi(x) = \zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} n^{-s} - \int_1^{\infty} t^{-s}\,\mathrm{d}t = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - t^{-s})\,\mathrm{d}t.$$

This last sum is an infinite sum of analytic functions. For positive real s it is the sum of small areas above the $y = t^{-s}$ curve but inside the circumscribing box of the curve over each unit interval, and hence it is bounded absolutely by 1. More generally, for complex s with positive real part we can quantify the smallness of the sum as follows. For all $t \in [n, n+1]$ we have

$$|n^{-s} - t^{-s}| = |s \int_n^t x^{-s-1}\,\mathrm{d}x| \leq |s| \int_n^t x^{-\mathrm{Re}(s)-1}\,\mathrm{d}x \leq |s| n^{-\mathrm{Re}(s)-1},$$

with the last quantity in the previous display independent of t and having the power of n smaller by 1. It follows that

$$\left| \int_n^{n+1} (n^{-s} - t^{-s})\,\mathrm{d}t \right| \leq |s| n^{-\mathrm{Re}(s)-1}.$$

This estimate shows that the sum $\psi(s) = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - t^{-s})\,\mathrm{d}t$ converges on $\{s : \mathrm{Re}(s) > 0\}$, uniformly on compact subsets, making $\psi(s)$ analytic there. Thus, in the relation

$$\zeta(s) = \psi(s) + \frac{1}{s-1}\,, \quad \mathrm{Re}(s) > 1,$$

the right side is meromorphic on $\mathrm{Re}(s) > 0$, its only singularity being a simple pole at $s = 1$ with residue 1. So the previous display extends $\zeta(s)$ to $\mathrm{Re}(s) > 0$ and gives it the same properties, as claimed.

The value $\psi(1) = \lim_{s\to 1}(\zeta(s) - \frac{1}{s-1})$ is called *Euler's constant* and denoted γ,

$$\zeta(s) = \frac{1}{s-1} + \gamma + \mathcal{O}(s-1), \qquad \gamma = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-1} - t^{-1})\,\mathrm{d}t.$$

With H_N denoting the Nth harmonic number $\sum_{n=1}^{N} n^{-1}$, Euler's constant is

$$\gamma = \lim_{N\to\infty} (H_N - \log N).$$

As above, this is the area above the $y = 1/x$ curve for $x \geq 1$ but inside the circumscribing boxes $[n, n+1] \times [0, 1/n]$ for $n \geq 1$.

The continuation argument just given should be viewed as a place-holder, because Riemann's deeper argument continues $\zeta(s)$ meromorphically to all of the complex plane and establishes a functional equation for the continuation.

With $\zeta(s)$ continued, its logarithmic derivative $\zeta'(s)/\zeta(s)$ also continues meromorphically to $\mathrm{Re}(s) > 0$, again having a simple pole at $s = 1$, this time with residue $\mathrm{res}_1(\zeta'(s)/\zeta(s)) = \mathrm{ord}_1\zeta(s) = -1$. Indeed, recall more generally that if a function f is meromorphic about c and not identically 0 then f'/f is again meromorphic about c with at most a simple pole at c, and

$$\mathrm{res}_c(f'/f) = \mathrm{ord}_c f.$$

The argument is that because $f(z) = (z-c)^m g(z)$ about c, with $m = \mathrm{ord}_c f$ and g nonzero at c,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-c} + \frac{g'(z)}{g(z)}, \qquad \frac{g'}{g} \text{ holomorphic about } c,$$

and so $\mathrm{res}_c(f'/f) = m$ as desired.

3.5. **Non-vanishing of zeta on** $\mathrm{Re}(s) = 1$**.** To help prove the next proposition, and for further use in section 3.7, compute that for $\mathrm{Re}(s) > 1$ the logarithmic derivative of $\zeta(s)$ is, from (2),

$$\frac{\zeta'(s)}{\zeta(s)} = (\log \zeta(s))' = -\sum_p \sum_{m \geq 1} \frac{\log p}{p^{ms}}. \tag{3}$$

(The coefficient function of the Dirichlet series in the previous display is the *von Mangoldt function*, $\Lambda(p^m) = \log p$ and $\Lambda(n) = 0$ if n is not a prime power.)

Proposition 3.1. $\zeta(s) \neq 0$ *for* $\mathrm{Re}(s) = 1$.

Proof. Fix any nonzero real t. Define

$$D(s) = \zeta(s)^3 \zeta(s+it)^4 \zeta(s+2it).$$

From the logarithmic derivative computation just above,

$$\frac{D'(s)}{D(s)} = -\sum_p \sum_{m \geq 1} \frac{\log p(3 + 4p^{-mit} + p^{-2mit})}{p^{ms}}.$$

We show that $0 \geq \mathrm{ord}_1 D(s)$, i.e., $D(s)$ is nonzero at 1. The order of vanishing is

$$\mathrm{ord}_1 D(s) = \mathrm{res}_1(D'(s)/D(s)) = \lim_{s \to 1^+} (s-1) D'(s)/D(s),$$

with s approaching 1 from the right on the real axis. Because this quantity is an integer it is real, and so it is the limit of $s - 1$ times the real part of $D'(s)/D(s)$,

$$\mathrm{ord}_1 D(s) = -\lim_{s \to 1^+} (s-1) \sum_p \sum_{m \geq 1} \frac{(3 + 4\cos(mt \log p) + \cos(2mt \log p)) \log p}{p^{ms}}.$$

But for any real θ,

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2\cos^2\theta - 1 = 2(1 + \cos\theta)^2 \geq 0,$$

and so the limit is nonpositive, i.e., $0 \geq \mathrm{ord}_1 D(s)$ as claimed. The result follows because $\mathrm{ord}_1 D(s) \geq -3 + 4\,\mathrm{ord}_1 \zeta(s+it)$, precluding the integer $\mathrm{ord}_1\zeta(s+it)$ from being positive. That is, $\zeta(1+it) \neq 0$. $\square$

3.6. **Improved continuation of the logarithmic derivative.** In consequence of $\zeta'(s)/\zeta(s)$ extending meromorphically from $\mathrm{Re}(s) > 1$ to $\mathrm{Re}(s) > 0$ with a simple pole at $s = 1$, and of $\zeta(s)$ never vanishing on $\mathrm{Re}(s) = 1$, also $(s-1)\zeta'(s)/\zeta(s)$ extends holomorphically from $\mathrm{Re}(s) > 1$ to $\mathrm{Re}(s) \geq 1$. Being holomorphic on $\mathrm{Re}(s) \geq 1$ and meromorphic on $\mathrm{Re}(s) > 0$, $(s-1)\zeta'(s)/\zeta(s)$ is in fact holomorphic on an open superset of $\mathrm{Re}(s) \geq 1$.

3.7. **Dominant term of the logarithmic derivative near $s = 1$.** For $\mathrm{Re}(s) > 1$, decompose the logarithmic derivative of $\zeta(s)$ in (3) into two terms, as in Euler's proof,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{p^s} - \sum_p \sum_{m\geq 2} \frac{\log p}{p^{ms}}\,, \quad \mathrm{Re}(s) > 1.$$

The second sum defines a holomorphic function on $\mathrm{Re}(s) > 1/2$ because its partial sums are entire and it converges uniformly on compacta therein. Indeed, $|p^{ms}| = p^{m\sigma}$ where $\sigma = \mathrm{Re}(s)$, and given $\sigma > 1/2$ there exists p_o such that $\log p < p^{\sigma-1/2}$ for all $p \geq p_o$; so, with $c = 1/(1-2^{-1/2}) = 2+\sqrt{2}$,

$$\sum_{\substack{p\geq p_o\\ m\geq 2}} \frac{\log p}{p^{m\sigma}} = \sum_{p\geq p_o} \frac{\log p}{(1-p^{-\sigma})p^{2\sigma}} < c\sum_{p\geq p_o} \frac{p^{\sigma-1/2}}{p^{2\sigma}} = c\sum_{p\geq p_o} \frac{1}{p^{\sigma+1/2}}\,.$$

This suffices to prove the uniform convergence.

The dominant term $-\sum_p \log p/p^s$ of $\zeta'(s)/\zeta(s)$ near $s = 1$ now takes the form $-D(s)$, where D is the Dirichlet-like series

$$D(s) = \sum_{n=1}^{\infty} \frac{c_n \log n}{n^s}\,, \qquad c_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Crucially, $\{c_n\}$ is the prime-indicator sequence that arose from the Chebyshev theta function and the prime-counting function in section 2.2. This series is holomorphic on the open right half plane $\mathrm{Re}(s) > 1$, and $(s-1)D(s)$ extends holomorphically to an open superset of the closed right half plane $\mathrm{Re}(s) \geq 1$, and $D(s)$ extends holomorphically to this set except for a simple pole at $s = 1$ with residue $\rho = 1$. Also, the condition $\vartheta(x) = \mathcal{O}(x)$ is already established. These will be precisely the hypotheses for the last result of this writeup, Corollary 4.2 below, whose conclusion is then that $\vartheta(x) \sim \rho x = x$, completing the proof of the Prime Number Theorem.

4. Convergence theorem, corollary on asymptotics, end of the proof

4.1. Theorem.

Theorem 4.1. *Consider a holomorphic function f on the open right half plane $\mathrm{Re}(s) > 0$, as follows: α is a bounded locally integrable function on $\mathbb{R}_{\geq 1}$, and f is the integral*

$$f(s) = \int_{t=1}^{\infty} \frac{\alpha(t)}{t^{s+1}}\,\mathrm{d}t.$$

Suppose that f extends to a holomorphic function on an open superset of the closed right half plane $\mathrm{Re}(s) \geq 0$. Then the integral that defines $f(s)$ for $\mathrm{Re}(s) > 0$ converges on the closed right half plane $\mathrm{Re}(s) \geq 0$.

The theorem also holds if instead $\{a_n\}$ is a bounded sequence of complex numbers and $f(s)$ is a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1}}\,,$$

by the same proof to follow.

Proof. It suffices to show that the integral converges at $s = 0$. Indeed, for any real y, the function $\tilde{f}(s) = f(s + iy)$ satisfies the same conditions as f, now with $\tilde{\alpha}(t) = \alpha(t)e^{-iyt}$ (or $\{\tilde{a}_n\} = \{a_n/n^{iy}\}$ in the Dirichlet series case), and the convergence at 0 of the integral that initially defines $\tilde{f}$ is precisely the convergence at iy of the integral that initially defines f.

For any $R \geq 1$ there exists $\delta = \delta_R > 0$ such that f is holomorphic on the compact region determined by the conditions $|s| \leq R$ and $\mathrm{Re}(s) \geq -\delta$, a truncated disk if $\delta < R$. Consider the counterclockwise boundary of this region, consisting of an arc determined by the conditions $|s| = R$ and $\mathrm{Re}(s) \geq -\delta$, and possibly a vertical segment determined by the conditions $|s| \leq R$ and $\mathrm{Re}(s) = -\delta$. Typically the arc will be less than a full circle and the vertical segment will be present. Let A and B respectively denote the portions of the boundary in the right and left half planes, so that the boundary is $A \cup B$ with A a right semicircle.

Let N be any positive integer. Because $f(0) = f(0)N^0$, Cauchy's integral representation and Cauchy's theorem give

$$2\pi i f(0) = \int_{A\cup B} f(s)N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) \mathrm{d}s. \tag{4}$$

Consider the Nth truncation of the integral for $f(s)$ on $\mathrm{Re}(s) > 0$,

$$f_N(s) = \int_{t=0}^{N} \alpha(t)e^{-st}\,\mathrm{d}t.$$

This is an entire function of s, and so we may express its value at 0 by integrating over the circle $A \cup -A$ rather than the truncated circle $A \cup B$,

$$2\pi i f_N(0) = \int_{A\cup -A} f_N(s)N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) \mathrm{d}s.$$

Further,

$$\int_{-A} f_N(s)N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) \mathrm{d}s = \int_{A} f_N(-s)N^{-s} \left(\frac{1}{s} + \frac{s}{R^2}\right) \mathrm{d}s,$$

and so in fact

$$2\pi i f_N(0) = \int_{A} f_N(s)N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) \mathrm{d}s + \int_{A} f_N(-s)N^{-s} \left(\frac{1}{s} + \frac{s}{R^2}\right) \mathrm{d}s. \tag{5}$$

Let $r_N = f - f_N$ denote the Nth remainder, a holomorphic function on an open superset of the closed right half plane $\mathrm{Re}(s) \geq 0$, represented by a tail integral on the open right half plane $\mathrm{Re}(s) > 0$. Proving the theorem amounts to showing that $\lim_N r_N(0) = 0$. Because $r_N = f - f_N$, the calculated expressions (4) and (5) for

$2\pi i f(0)$ and $2\pi i f_N(0)$ give

$$\begin{aligned}2\pi i r_N(0) &= \int_A f(s)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s + \int_B f(s)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s \\ &\quad - \int_A f_N(s)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s - \int_A f_N(-s)N^{-s}\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s,\end{aligned}$$

which rearranges to give $2\pi i r_N(0)$ as a sum of three terms,

$$\begin{aligned}2\pi i r_N(0) &= \int_A \left(r_N(s)N^s - f_N(-s)N^{-s}\right)\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s \\ &\quad + \int_{B\cap\{|s|=R\}} f(s)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s \\ &\quad + \int_{B\cap\{\mathrm{Re}(s)=-\delta\}} f(s)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s.\end{aligned} \tag{6}$$

Next compute some estimates. Let b be a bound of the function α, so that b depends only on f. Let σ denote the real part of s.

- For $\mathrm{Re}(s) > 0$, with b as just above,
  $$|r_N(s)| \le \frac{b}{\sigma N^\sigma}\,.$$
  Indeed, $|r_N(s)| = \left|\int_{t=N}^\infty \alpha(t)t^{-s-1}\,\mathrm{d}t\right| \le b\int_{t=N}^\infty t^{-\sigma-1}\,\mathrm{d}t = b/(\sigma N^\sigma)$. (In the Dirichlet series case, with $r_N(s) = \sum_{n=N}^\infty a_n n^{-s-1}$, the upper bound is $b(1/N^{\sigma+1} + 1/(\sigma N^\sigma).)$
- For $\mathrm{Re}(s) > 0$, with b as above,
  $$|f_N(-s)| \le \frac{bN^\sigma}{\sigma}\,.$$
  Indeed, $|f_N(-s)| = \left|\int_{t=1}^N \alpha(t)t^{s-1}\,\mathrm{d}t\right| \le b\int_{t=1}^N t^{\sigma-1}\,\mathrm{d}t = b(N^\sigma - 1)/\sigma$.
- For $|s| = R$,
  $$\frac{1}{s}+\frac{s}{R^2} = \frac{2\sigma}{R^2}\,.$$
  Indeed, $s = Re^{i\theta}$, and so $s^{-1} + sR^{-2} = (e^{i\theta}+e^{-i\theta})R^{-1} = 2R\cos\theta\cdot R^{-2}$.
- For s on the vertical segment portion of B, because $\sigma = -\delta$ and $|s| \le R$,
  $$\left|\frac{1}{s}+\frac{s}{R^2}\right| \le \frac{1}{\delta}+\frac{1}{R} = \frac{R+\delta}{R\delta}\,.$$

From the first three estimates and from A having length πR, the first term of $2\pi i r_N(0)$ in (6) satisfies

$$\left|\int_A \left(r_N(s)N^s - f_N(-s)N^{-s}\right)\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s\right| \le \frac{4\pi b}{R}\,.$$

Let $\varepsilon > 0$ be given. If $R > 4\pi b/\varepsilon$ then $4\pi b/R < \varepsilon$.

For the given $\varepsilon > 0$, and with $R > 4\pi b/\varepsilon$ fixed, take a compatible $\delta = \delta_R > 0$, freely stipulating that $\delta < 1$, such that f is holomorphic on and inside $A \cup B$. Let M bound f on this compact region. The second term of $2\pi i r_N(0)$ in (6) satisfies, again using the first three estimates, the conditions on δ, and a little geometry,

$$\left|\int_{B\cap\{|s|=R\}} f(s)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right)\mathrm{d}s\right| \le \frac{8M}{R^2}\int_{\sigma=-\delta}^0 N^\sigma\,\mathrm{d}\sigma < \frac{8M}{R^2\log N}\,.$$

If $N > \exp(8M/(R^2\varepsilon))$ then $8M/(R^2 \log N) < \varepsilon$.

Still with ε and R and δ, the fourth estimate and the fact that $B \cap \{\mathrm{Re}(s) = -\delta\}$ has length at most $2R$ show that the third term of $2\pi i r_N(0)$ in (6) satisfies

$$\left| \int_{B\cap\{\mathrm{Re}(s)=-\delta\}} f(s) N^s \left(\frac{1}{s} + \frac{s}{R^2} \right) \mathrm{d}s \right| \leq \frac{2M(R+\delta)}{\delta N^\delta} .$$

If $N > (2M(R+\delta)/(\delta\varepsilon))^{1/\delta}$ then $2M(R+\delta)/(\delta N^\delta) < \varepsilon$.

Altogether, given $\varepsilon > 0$, take $R > 4\pi b/\varepsilon$ and suitable $\delta = \delta_R < 1$, and then $|2\pi i r_N(0)| < 3\varepsilon$ for all large enough N. Thus $\{r_N(0)\}$ converges to 0, which is to say that the integral that defines $f(s)$ for $\mathrm{Re}(s) > 0$ converges at $s = 0$ to $f(0)$. □

4.2. **Corollary.** The next result follows from the previous theorem.

Corollary 4.2. *Let $\{c_n\}$ be a sequence of nonnegative real numbers such that the sum*

$$D(s) = \sum_{n=1}^{\infty} \frac{c_n \log n}{n^s}$$

is holomorphic on the open right half plane $\mathrm{Re}(s) > 1$. Suppose that $(s-1)D(s)$ extends holomorphically to an open superset of the closed right half plane $\mathrm{Re}(s) \geq 1$, so that $D(s)$ extends holomorphically to this set except possibly for a simple pole at $s = 1$. Let

$$\rho = \mathrm{res}_1 D(s).$$

Suppose that the function

$$S(x) = \sum_{n \leq x} c_n \log n \quad \textit{for real } x \geq 1$$

satisfies

$$S(x) = \mathcal{O}(x).$$

Then

$$S(x) \sim \rho x.$$

Proof. For $\mathrm{Re}(s) > 1$, write $D(s)$ as a Stieltjes integral, integrate by parts with the boundary terms $S(t)/t^s|_1^\infty$ vanishing,

$$D(s) = \int_1^\infty \frac{\mathrm{d}S(t)}{t^s} = s \int_1^\infty \frac{S(t)}{t^{s+1}} \, \mathrm{d}t.$$

Consequently for $\mathrm{Re}(s) > 0$, recalling the quantity $\rho = \mathrm{res}_1 D(s)$,

$$\int_{t=1}^\infty \frac{S(t)/t - \rho}{t^{s+1}} \, \mathrm{d}t = \frac{D(s+1)}{s+1} - \frac{\rho}{s} .$$

Because $D(s+1)/(s+1) \sim \rho/(s(s+1)) = \rho/s - \rho/(s+1)$ for s near 0, the right side extends holomorphically to an open superset of the closed right half plane $\mathrm{Re}(s) \geq 0$, and hence so does the left side. Further, the function $S(t)/t - \rho$ is bounded and locally integrable on $\mathbb{R}_{\geq 1}$, so it meets the conditions on the α in the convergence theorem. The theorem says that the integral on the left side converges for $\mathrm{Re}(s) \geq 0$, and in particular for $s = 0$. That is,

$$\int_{t=1}^\infty \frac{S(t) - \rho t}{t^2} \, \mathrm{d}t \quad \text{converges.}$$

This convergence and the fact that $S(x)$ is nonnegative and increasing show that $S(x) \sim \rho x$, meaning that $\lim_{x\to\infty} S(x)/x = \rho$, as follows. Let $\varepsilon > 0$ be given. Suppose that $S(x) \geq (1+\varepsilon)\rho x$ for a sequence of x-values going to ∞. Estimate that for such x,

$$\int_{t=x}^{(1+\varepsilon)x} \frac{S(t)-\rho t}{t^2}\,\mathrm{d}t \geq \int_{t=x}^{(1+\varepsilon)x} \frac{(1+\varepsilon)\rho x-\rho t}{t^2}\,\mathrm{d}t = \rho\int_{t=1}^{1+\varepsilon} \frac{1+\varepsilon-t}{t^2}\,\mathrm{d}t,$$

the last quantity positive and independent of x. This contradicts the convergence of the integral. Similarly, now freely taking $\varepsilon < 1$, if $S(x) \leq (1-\varepsilon)\rho x$ for a sequence of x-values going to ∞ then for such x,

$$\int_{t=(1-\varepsilon)x}^{x} \frac{S(t)-\rho t}{t^2}\,\mathrm{d}t \leq \rho\int_{t=1-\varepsilon}^{1} \frac{1-\varepsilon-t}{t^2}\,\mathrm{d}t,$$

negative and independent of x, again violating convergence. □

4.3. **End of the proof.** As noted at the end of section 3.7, the case where $c_n = 1$ if n is prime and $c_n = 0$ otherwise completes the proof of the Prime Number Theorem. In this case, $D(s)$ is (minus) the dominant term of the logarithmic derivative $\zeta'(s)/\zeta(s)$, with residue $\rho = 1$ at $s = 1$, and $S(x)$ is the Chebyshev theta function $\vartheta(x)$, known to be $\mathcal{O}(x)$. The asymptotic result $\vartheta(x) \sim x$ from Corollary 4.2 is exactly what is needed to finish the Prime Number Theorem argument.