

MORERA'S THEOREM

Let Ω be a region. Recall some ideas:

- Cauchy's theorem says that if γ is a simple closed rectifiable curve in Ω , and if f is an analytic function on an open superset of γ and its interior, then

$$\int_{\gamma} f(z) dz = 0.$$

- In consequence of Cauchy's theorem, Cauchy's integral representation formula says that if γ is a simple closed rectifiable curve in Ω , and if f is an analytic function on an open superset of γ and its interior, then for every point z in the interior of γ ,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}.$$

- Differentiation under the integral sign shows that if a continuous function $f : \Omega \rightarrow \mathbb{C}$ has the integral representation of the previous bullet, then f is \mathcal{C}^{∞} on Ω , and its derivatives also have integral representation; specifically, for any γ and z as in the previous bullet,

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}, \quad k = 0, 1, 2, \dots$$

In particular, an analytic function on Ω is \mathcal{C}^{∞} on Ω . From here, a geometric series argument shows that in fact an analytic function on Ω is \mathcal{C}^{ω} on Ω .

1. STATEMENT

Morera's theorem is a partial converse of Cauchy's theorem, as follows.

Theorem 1.1 (Morera). *Let Ω be a region, and let $f : \Omega \rightarrow \mathbb{C}$ be continuous. Suppose that*

$$\int_{\gamma} f(z) dz = 0 \quad \text{for all simple closed rectifiable curves } \gamma \text{ in } \Omega.$$

Then f is analytic on Ω .

Proof. We need to show that f' exists on Ω . Fix any point z_0 in Ω . The following function is well defined:

$$F : \Omega \rightarrow \mathbb{C}, \quad F(z) = \int_{z_0}^z f(\zeta) d\zeta,$$

where the integral is taken along any rectifiable curve from z_0 to z . For any $z \in \Omega$ and all small enough nonzero $h \in \mathbb{C}$ we have, integrating over the line segment from

z to $z + h$,

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{\int_z^{z+h} f(\zeta) d\zeta}{h} \\ &= \frac{\int_z^{z+h} (f(z) + o(1)) d\zeta}{h} \\ &= f(z) + \frac{1}{h} \int_z^{z+h} o(1) d\zeta. \end{aligned}$$

Given any $\varepsilon > 0$, the integrand satisfies $|o(1)| \leq \varepsilon$ if h is small enough, and so the absolute value of the integral is at most $\varepsilon|h|$ for all such h . That is, for every $\varepsilon > 0$, the difference quotient is within ε of $f(z)$ for all small enough nonzero h . This means precisely that $F'(z)$ exists and equals $f(z)$. Because z is any point of Ω this shows that

$$F' = f \quad \text{on } \Omega.$$

Because F is analytic on Ω , it is \mathcal{C}^∞ on Ω , and in particular its second derivative exists on Ω . That is, f' exists on Ω . \square

The proof of Morera's theorem shows that if $\int_\gamma f(z) dz = 0$ for all simple closed rectifiable curves γ in Ω then $f = F'$ for some analytic $F : \Omega \rightarrow \mathbb{C}$. The converse of this statement is true as well, by the complex fundamental theorem of calculus, and so we have a partial converse to Morera's theorem, that if an analytic function f is a derivative then $\int_\gamma f(z) dz = 0$ for all simple closed rectifiable curves γ in Ω . However, the full converse of Morera's theorem is not true, the function $f(z) = 1/z$ on $\Omega = \mathbb{C} - \{0\}$ being the standard counterexample. Although f is analytic on Ω , it is not a derivative there, and its integral over the unit circle is nonzero.

2. CONSEQUENCE: THE CONVERSE OF CAUCHY'S THEOREM

By contrast, an essential converse of Cauchy's theorem is true, thanks to Morera's theorem. Again let Ω be a region, and let $f : \Omega \rightarrow \mathbb{C}$ be continuous. Cauchy's theorem says that if $f : \Omega \rightarrow \mathbb{C}$ is analytic then $\int_\gamma f(z) dz = 0$ for all simple closed rectifiable curves γ in Ω such that the interior of γ lies in Ω .

Now rather than assume that f is analytic, assume instead that $\int_\gamma f(z) dz = 0$ for all simple closed rectifiable curves γ in Ω such that the interior of γ lies in Ω . For each point z of Ω , let B_z denote the largest open disk about z in Ω , and note that every simple closed rectifiable curve γ in B_z is such that the interior of γ lies in B_z , and so $\int_\gamma f(z) dz = 0$. Thus f on B_z satisfies the hypothesis of Morera's theorem, and so Morera's theorem says that f' exists on B_z . In particular $f'(z)$ exists. Because z is an arbitrary point of Ω , this shows that f is analytic on Ω .