MATH 311: COMPLEX ANALYSIS — CONTOUR INTEGRALS LECTURE

Recall the Residue Theorem: *Let γ be a simple closed loop, traversed counterclockwise. Let f be a function that is analytic on γ and meromorphic inside γ. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{c \text{ inside } \gamma} \mathrm{Res}_c(f).$$

This writeup shows how the Residue Theorem can be applied to integrals that arise with no reference to complex analysis.

1. Computing Residues

Proposition 1.1. *Let f have a simple pole at c. Then*

$$\mathrm{Res}_c(f) = \lim_{z\to c}(z-c)f(z).$$

Proposition 1.2. *Let f have a pole of order $n \geq 1$ at c. Define a modified function $g(z) = (z-c)^n f(z)$. Then*

$$\mathrm{Res}_c(f) = \frac{1}{(n-1)!}\lim_{z\to c} g^{(n-1)}(z).$$

Proof. It suffices to prove the second proposition, since it subsumes the first. Recall that the residue is the the coefficient a_{-1} of $1/(z-c)$ in the Laurent series. The proof is merely a matter of inspection:

$$f(z) = \frac{a_{-n}}{(z-c)^n} + \cdots + \frac{a_{-1}}{z-c} + a_0 + \cdots,$$

and so the modified function g is

$$g(z) = a_{-n} + \cdots + a_{-1}(z-c)^{n-1} + a_0(z-c)^n + \cdots,$$

whose $(n-1)$st derivative is

$$g^{(n-1)}(z) = (n-1)!\,a_{-1} + n(n-1)\cdots 2\,a_0(z-c) + \cdots.$$

Thus

$$\lim_{z\to c} g^{(n-1)}(z) = g^{(n-1)}(c) = (n-1)!\,a_{-1}.$$

This is the desired result. □

In applying the propositions, we do *not* go through these calculations again every time. L'Hospital's Rule will let us take the limits without computing the Laurent series. Incidentally, note that a slight rearrangement of the proposition,

$$\mathrm{Res}_c\left(\frac{g(z)}{(z-c)^{n+1}}\right) = \frac{g^{(n)}(c)}{n!}, \quad n \geq 0,$$

shows that the Residue Theorem subsumes Cauchy's integral representation for derivatives.

2. Rational Functions

Let $f(x) = p(x)/q(x)$ be a rational function of the real variable x, where $q(x) \neq 0$ for all $x \in \mathbb{R}$. Further assume that

$$\deg(f) \leq -2,$$

meaning that $\deg(q) \geq \deg(p) + 2$. Then

$$\int_{x=-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\mathrm{Im}(c)>0} \mathrm{Res}_c(f).$$

Here is an example. Let

$$f(z) = \frac{z^2}{z^4+1}\,.$$

We want to compute the integral

$$I = \int_{x=-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

To do so, let r be a large positive number, and let γ consist of two pieces: the segment $[-r, r]$ of the real axis and the upper half-circle γ_r of radius r. Thus

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{x=-r}^{r} f(x)\,\mathrm{d}x + \int_{\gamma_r} f(z)\,\mathrm{d}z.$$

On the half-circle γ_r we have

$$|f(z)| \sim \frac{1}{r^2} \quad (\text{since } \deg(f) = -2),$$

while the length of γ_r is πr. Therefore, letting the symbol "$\preceq$" mean "is asymptotically at most,"

$$\left|\int_{\gamma_r} f(z)\,\mathrm{d}z\right| \preceq \frac{\pi r}{r^2} \overset{r\to\infty}{\longrightarrow} 0.$$

On the other hand,

$$\int_{x=-r}^{r} f(x)\,\mathrm{d}x \overset{r\to\infty}{\longrightarrow} I,$$

and so

$$\int_\gamma f(z)\,\mathrm{d}z \overset{r\to\infty}{\longrightarrow} I.$$

But for all large enough values of r, also

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{\mathrm{Im}(c)>0} \mathrm{Res}_c(f).$$

So to evaluate the integral I we need only compute the sum of the residues.

The denominator z^4+1 of $f(z)$ shows that f has simple poles at the fourth roots of -1. These values are

$$\zeta_8, \quad \zeta_8^3, \quad \zeta_8^5, \quad \zeta_8^7, \qquad \text{where } \zeta_8 = e^{2\pi i/8}.$$

Of the four roots, only ζ_8 and ζ_8^3 lie in the upper half plane. Compute, using L'Hospital's Rule that

$$\lim_{z\to\zeta_8} \frac{(z-\zeta_8)z^2}{z^4+1} = \lim_{z\to\zeta_8} \frac{z^2+(z-\zeta_8)2z}{4z^3} = \lim_{z\to\zeta_8} \frac{z^2}{4z^3} = \lim_{z\to\zeta_8} \frac{1}{4z} = \frac{1}{4\zeta_8} = \frac{\zeta_8^7}{4},$$

and similarly

$$\lim_{z\to\zeta_8^3}\frac{(z-\zeta_8^3)z^2}{z^4+1}=\frac{1}{4\zeta_8^3}=\frac{\zeta_8^5}{4}.$$

So the sum of the residues is

$$\sum_{\mathrm{Im}(c)>0}\mathrm{Res}_c(f)=\frac{\zeta_8^5+\zeta_8^7}{4}=\frac{-\sqrt{2}\,i}{4}=\frac{-i}{2\sqrt{2}}.$$

And consequently the integral is

$$I=2\pi i\frac{-i}{2\sqrt{2}}=\boxed{\frac{\pi}{\sqrt{2}}}.$$

3. Rational Functions Times Sine or Cosine

Consider the integral

$$I=\int_{x=0}^{\infty}\frac{\sin x}{x}\,\mathrm{d}x.$$

To evaluate this real integral using the residue calculus, define the complex function

$$f(z)=\frac{e^{iz}}{z}.$$

This function is meromorphic on $\mathbb{C}$, with its only pole being a simple pole at the origin.

Let r be a large positive real number, and let ε be a small positive real number. Define a contour γ consisting of four pieces:

$$\gamma=[-r,-\varepsilon]\cup\gamma_\varepsilon\cup[\varepsilon,r]\cup\gamma_r,$$

where γ_ε is the upper half-circle of radius ε, traversed clockwise, while γ_r is the upper half-circle of radius r, traversed counterclockwise. By the Residue Theorem (which subsumes Cauchy's Theorem),

$$\int_\gamma f(z)\,\mathrm{d}z=0.$$

Note that on γ_ϵ we have

$$f(z)\sim\frac{e^0}{z}=\frac{1}{z},$$

so that, since γ_ϵ is a *half*-circle traversed *clockwise*,

$$\int_{\gamma_\varepsilon}f(z)\,\mathrm{d}z\sim-\pi i,$$

and this approximation tends to equality as ε shrinks toward 0.

Meanwhile, parametrize γ_r by letting $z=re^{i\theta}$ where $0\le\theta\le\pi$. On γ_r we have

$$|f(z)|=|f(re^{i\theta})|=\left|\frac{\exp(ire^{i\theta})}{re^{i\theta}}\right|=\frac{|\exp(ir(\cos\theta+i\sin\theta))|}{r}=\frac{e^{-r\sin\theta}}{r},$$

and $|\mathrm{d}z|=|ire^{i\theta}\,\mathrm{d}\theta|=r\,\mathrm{d}\theta$, so that

$$\left|\int_{\gamma_r}f(z)\,\mathrm{d}z\right|\le\int_{\theta=0}^{\pi}e^{-r\sin\theta}\,\mathrm{d}\theta.$$

The integrand bounded above by 1 (and now matter how large r gets, the integrand always equals 1 for $\theta=0$ and $\theta=\pi$), but as r grows, the integrand tends to 0 uniformly on any any compact subset of $(0,\pi)$, and so the integral goes to 0 as r goes to ∞.

The remaining two pieces of the integral are, by a change of variable,

$$\begin{aligned}\int_{x=-r}^{-\varepsilon} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{x=\varepsilon}^{r} \frac{e^{ix}}{x}\,\mathrm{d}x &= \int_{x=r}^{\varepsilon} \frac{e^{-ix}}{-x}\,\mathrm{d}(-x) + \int_{x=\varepsilon}^{r} \frac{e^{ix}}{x}\,\mathrm{d}x \\ &= \int_{x=\varepsilon}^{r} \frac{e^{ix}-e^{-ix}}{x}\,\mathrm{d}x \\ &= 2i\int_{x=\varepsilon}^{r} \frac{\sin x}{x}\,\mathrm{d}x.\end{aligned}$$

Now let $\varepsilon \to 0^+$ and let $r \to \infty$. Our calculation has shown that $2iI - \pi i = 0$, i.e.,

$$I = \boxed{\frac{\pi}{2}}.$$

4. Rational Functions of Cosine and Sine

Consider the integral

$$I = \int_{\theta=0}^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}, \quad a > 1.$$

This integral is not improper, i.e., its limits of integration are finite. The distinguishing characteristic here is that the integrand is a rational function of $\cos\theta$ and $\sin\theta$, integrated from 0 to 2π. Thus we may set

$$z = e^{i\theta}, \quad 0 \le \theta \le 2\pi,$$

and view the integral as a contour integral over the unit circle. On the unit circle we have

$$\cos\theta = \frac{z+z^{-1}}{2}, \qquad \sin\theta = \frac{z-z^{-1}}{2i}, \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}.$$

Thus the integral becomes the integral of a rational function of z over the unit circle, and the new integral can be computed by the residue calculus. For the particular integral in question, the calculation is

$$I = \int_{|z|=1} \frac{1}{a+\frac{z+z^{-1}}{2}} \cdot \frac{\mathrm{d}z}{iz} = \frac{2}{i}\int_{|z|=1} \frac{\mathrm{d}z}{z^2+2az+1}.$$

Analyze the denominator as follows:

$$z^2 + 2az + 1 = (z-r_1)(z-r_2), \quad r_1 + r_2 = -2a,\ r_1 r_2 = 1.$$

Neither root lies on the unit circle since the condition $a > 1$ ensures that the original denominator $a + \cos\theta$ is never zero. Let r_1 be the root inside the circle and r_2 be the root outside it. Thus the integral is

$$I = \frac{2}{i} \cdot 2\pi i \operatorname{Res}_{r_1}\left(\frac{1}{(z-r_1)(z-r_2)}\right) = 4\pi \lim_{z\to r_1} \frac{(z-r_1)}{(z-r_1)(z-r_2)} = \frac{4\pi}{(r_1-r_2)}.$$

But the roots of the quadratic polynomial $z^2 + 2az + 1$ are

$$r_1, r_2 = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1},$$

and so

$$I = \frac{4\pi}{(r_1 - r_2)} = \boxed{\frac{2\pi}{\sqrt{a^2-1}}}.$$

5. Integration Around a Branch Point

Consider the integral

$$I = \int_{x=0}^{\infty} \frac{x^{-s}}{x+1}\,\mathrm{d}x\,, \quad 0 < s < 1.$$

For positive real values x we have the formula

$$x^{-s} = \exp(-s\ln x).$$

Let ε be a small positive real number and r be a large positive real number. Define a contour consisting of four pieces:

$$\gamma = \gamma_+ \cup \gamma_r \cup \gamma_- \cup \gamma_\varepsilon,$$

where

- γ_+ is the the real axis traversed from ε up to r, viewing its points as having argument 0,
- γ_r is the circle of radius r traversed counterclockwise, viewing its points as having argument increasing from 0 to 2π,
- γ_- is the the real axis traversed from r down to ε, viewing its points as having argument 2π,
- and γ_ε is the circle of radius ε traversed clockwise, viewing its points as having argument decreasing from 2π to 0.

On γ_r we have

$$|z^{-s}| = |\exp(-s\ln r - is\theta)| = r^{-s},$$

so that, since $|z+1| \sim r$ on γ_r,

$$\left|\frac{z^{-s}}{z+1}\right| \preceq \frac{r^{-s}}{r}\,.$$

Also, γ_r has length $2\pi r$, and so

$$\left|\int_{\gamma_r} \frac{z^{-s}}{z+1}\,\mathrm{d}z\right| \preceq 2\pi r^{-s} \overset{r\to\infty}{\longrightarrow} 0.$$

Here it is relevant that $s > 0$. A similar analysis, using the condition $1 - s > 0$ and the fact that $|z+1| \sim 1$ on γ_ε shows that also

$$\left|\int_{\gamma_\varepsilon} \frac{z^{-s}}{z+1}\,\mathrm{d}z\right| \preceq 2\pi\varepsilon^{1-s} \overset{\varepsilon\to 0^+}{\longrightarrow} 0.$$

For points $z = x$ on γ_+ we have

$$z^{-s} = x^{-s},$$

but for points $z = x$ on γ_-, where we are viewing the argument as 2π rather than 0, we have

$$z^{-s} = \exp(-s\log z) = \exp(-s\ln|z| - 2\pi is) = x^{-s}e^{-2\pi is}.$$

And so

$$\left(\int_{\gamma_+} + \int_{\gamma_-}\right) \frac{z^{-s}}{z+1}\,\mathrm{d}z = (1 - e^{-2\pi is}) \int_{x=\varepsilon}^{r} \frac{x^{-s}}{x+1}\,\mathrm{d}x$$

The coefficient in front of the integral is

$$1 - e^{-2\pi i s} = (e^{\pi i s} - e^{-\pi i s})e^{-\pi i s} = 2i\sin(\pi s)\, e^{-\pi i s},$$

and so as $\varepsilon \to 0^+$ and $r \to \infty$, we have

$$\int_\gamma \frac{z^{-s}}{z+1}\,\mathrm{d}z \longrightarrow 2i\sin(\pi s)e^{-\pi i s}\, I.$$

But also,

$$\begin{aligned}\int_\gamma \frac{z^{-s}}{z+1}\,\mathrm{d}z &= 2\pi i\,\mathrm{Res}_{-1}\left(\frac{z^{-s}}{z+1}\right)\\ &= 2\pi i(-1)^{-s}\\ &= 2\pi i\exp(-s\ln|-1| - is\pi)\\ &= 2\pi i e^{-\pi i s}.\end{aligned}$$

In sum,

$$2i\sin(\pi s)e^{-\pi i s} I = 2\pi i e^{-\pi i s},$$

so that

$$I = \boxed{\frac{\pi}{\sin \pi s}}.$$

6. The Riemann Zeta Function for Even Integers

The Riemann zeta function is

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \mathrm{Re}(s) > 1.$$

To evaluate $\zeta(k)$ where $k \geq 2$ is an even integer, we use the meromorphic function

$$f(z) = \pi \cot \pi z.$$

This function has a simple pole with residue 1 at $z = 0$ because for z near 0,

$$f(z) \sim \frac{1}{z}.$$

Thus, by $\mathbb{Z}$-periodicity, f has a simple pole with residue 1 at each integer.

Let n be a positive integer. Let γ be the rectangle with vertical sides at $\pm(n+1/2)$ and with horizontal sides at $\pm in$. For any even integer $k \geq 2$ we have

$$\int_\gamma \frac{\pi\cot\pi z}{z^k}\,\mathrm{d}z = 2\pi i\left(\mathrm{Res}_0\left(\frac{\pi\cot\pi z}{z^k}\right) + 2\sum_{m=1}^{n}\frac{1}{m^k}\right).$$

But

$$\pi\cot\pi z = \pi i\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = \pi i + \frac{2\pi i}{e^{2\pi i z}-1}\,,$$

while from the homework we know that

$$\frac{2\pi i z}{e^{2\pi i z}-1} = \sum_{j=0}^{\infty}\frac{B_j}{j!}(2\pi i z)^j,$$

with $B_k = 0$ for all odd k except for $B_1 = -1/2$. And so

$$\pi\cot\pi z = \sum_{j=0}^{\infty}\frac{B_j}{j!}(2\pi i)^j z^{j-1}, \quad \text{summing only over even } j.$$

Therefore,

$$\operatorname{Res}_0\left(\frac{\pi\cot\pi z}{z^k}\right) = \frac{(2\pi i)^k B_k}{k!} \quad \text{for even } k \geq 2.$$

Summarizing so far, the integral is

$$\int_\gamma \frac{\pi\cot\pi z}{z^k}\,\mathrm{d}z = 2\pi i\left(\frac{(2\pi i)^k B_k}{k!} + 2\sum_{m=1}^{n}\frac{1}{m^k}\right) \overset{n\to\infty}{\longrightarrow} 2\pi i\left(\frac{(2\pi i)^k B_k}{k!} + 2\zeta(k)\right).$$

But also, the integral tends to 0 as n gets large. To see this, estimate the integrand $\pi\cot\pi z/z^k$ on γ, by returning to the formula

$$\pi\cot\pi z = \pi i + \frac{2\pi i}{e^{2\pi i z} - 1}.$$

If $z = \pm(n + 1/2) + iy$ then

$$e^{2\pi i z} - 1 = -e^{-2\pi y} - 1 < -1,$$

and so the formula shows that (small exercise)

$$\text{if } z = \pm(n + 1/2) + iy \text{ then } |\pi\cot\pi z| < \pi.$$

On the other hand, if $z = x \pm in$ and n is large then $|e^{2\pi i z}| = e^{\pm 2\pi n}$ is either very large or very close to 0, and in either case

$$\text{if } z = x \pm in \text{ then } |\pi\cot\pi z| \sim \pi.$$

Thus two conditions hold on γ,

$$|\pi\cot\pi z| \preceq \pi \quad \text{and} \quad \left|\frac{1}{z^k}\right| \preceq \frac{1}{n^2}.$$

Since γ has length roughly $8n$, it follows that

$$\int_\gamma \frac{\pi\cot\pi z}{z^k}\,\mathrm{d}z \overset{n\to\infty}{\longrightarrow} 0.$$

By the explicit formula for the integral, it follows that

$$\boxed{2\zeta(k) = -\frac{(2\pi i)^k B_k}{k!} \quad \text{for even } k \geq 2.}$$

That is,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945},$$

and so on.

7. The Fourier Transform of the Gaussian

The one-dimensional Gaussian function is the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = e^{-\pi x^2}.$$

An exercise in multivariable calculus shows that f is normalized,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1.$$

Its Fourier transform is the function

$$\widehat{f} : \mathbb{R} \longrightarrow \mathbb{R}, \quad \widehat{f}(\eta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\eta x}\,\mathrm{d}x.$$

(The Fourier transform of the Gaussian is real-valued because the Gaussian is even and the sine function is odd, so that the imaginary part of the integral vanishes.) We use contour integration to show that $\hat{f} = f$. Compute that the integrand is

$$f(x)e^{-2\pi i\eta x} = e^{-\pi(x^2+2i\eta x-\eta^2)}e^{-\pi\eta^2} = e^{-\pi(x+i\eta)^2}f(\eta),$$

and so the Fourier transform of f is in fact

$$\widehat{f}(\eta) = f(\eta)\int_{-\infty}^{\infty} e^{-\pi(x+i\eta)^2}\,\mathrm{d}x.$$

It suffices to show the integral in the previous display is 1, and to show this, it suffices to show that the integral is the integral of the original Gaussian, since the original Gaussian is normalized.

To show that the integral in the previous display is the integral of the Gaussian, let γ be the rectangular contour that traverses from $-r$ to r along the real axis, then up to $r + i\eta$, then horizontally back to $-r + i\eta$, and finally back down to $-r$. Let

$$f(z) = e^{-\pi z^2}.$$

Since f is entire, the Residue Theorem says that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Also, the integrals along the sides of the rectangle go to 0 as r gets large. This is because if $z = \pm r + iy$ for any y between 0 and η then

$$|f(z)| = |e^{-\pi(\pm r+iy)^2}| = |e^{-\pi(r^2\pm 2iry-y^2)}| = e^{-\pi(r^2-y^2)},$$

and so as soon as $r \gg |\eta|$, the integrand is uniformly small as y varies from 0 to η.

Since the total integral vanishes and the side integrals go to 0 as r grows, the top and bottom integrals agree in the limit as $r \to \infty$. But the top integral is the integral that we want to equal 1,

$$\int_{-\infty+i\eta}^{\infty+i\eta} e^{-\pi z^2}\,\mathrm{d}z = \int_{-\infty}^{\infty} e^{-\pi(x+i\eta)^2}\,\mathrm{d}x,$$

while the bottom integral is the original Gausssian integral, which does equal 1,

$$\int_{-\infty}^{\infty} e^{-\pi z^2}\,\mathrm{d}z = \int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x = 1.$$

Thus the Fourier transform equals the original Gaussian, as claimed. That is,

$$\boxed{\widehat{f} = f \quad \text{for } f(x) = e^{-\pi x^2}.}$$

8. An Extraction Integral

Let x be a positive real number, and let σ also be a positive real number. We show that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s}\,\mathrm{d}s = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } 0 < x < 1, \end{cases} \qquad \sigma > 0.$$

The idea of the proof is that if $x > 1$ then the vertical line of integration slides to the left, picking up a residue at zero, until the integral vanishes, and if $0 < x < 1$ then similarly the line slides to the right, not picking up a residue. We make this precise.

Let $f(s) = x^s/s$, a meromorphic function on $\mathbb{C}$ whose only pole is a simple pole at $s = 0$ with residue 1. Let $\varepsilon > 0$ be given.

Assume first that $x > 1$. Let a and b be large positive real numbers. Consider the rectangle that proceeds counterclockwise from $\sigma - ib$ to $\sigma + ib$ to $-a + ib$ to $-a - ib$ and back to $\sigma - ib$. The integral of $f(z)$ about this rectangle is $2\pi i$, and the integral up the right side of the rectangle goes to the integral in the previous display as b goes to ∞. Integrating down the left side of the rectangle, we have $f(z) = \mathcal{O}(x^{-a}/a)$, and that side has length is $\mathcal{O}(b)$. So if $x^{-a}b/a < \varepsilon$ then the integral down the left side of the rectangle is small. On the top and bottom of the rectangle, we have $f(z) = \mathcal{O}(1/b)$, and those sides have length $\mathcal{O}(a)$. So if $a/b < \varepsilon$ then the integrals along the top and bottom of the rectangle are small. Choose a large enough that $1/\varepsilon < \varepsilon x^a$, here using the condition that $x > 1$, and then choose any b such that $1/\varepsilon < b/a < \varepsilon x^a$. These choices give the conditions $x^{-a}b/a < \varepsilon$ and $a/b < \varepsilon$, and the contour integral integral is close to the integral in the previous display.

The argument for $0 < x < 1$ is similar, but with the rectangle going far to the right rather than far to the left, and this time picking up no residue.