MATH 311: COMPLEX ANALYSIS — INTEGRATION LECTURE

Contents

1. Introduction

The data for a complex path integral

$$\int_\gamma f(z)\,\mathrm{d}z$$

and for a related integral

$$\int_\gamma f(z)\,|\mathrm{d}z|$$

are as follows.

- $\Omega \subset \mathbb{C}$ is a region,
- $f : \Omega \longrightarrow \mathbb{C}$ is a continuous function,
- $\gamma : [a, b] \longrightarrow \Omega$ is a continuous path.

However, the assumption that the path γ is continuous is not strong enough to guarantee that these integrals are sensible. This writeup discusses two ways to address this issue, and how they relate.

The first approach to complex path integrals is that

γ is assumed to be piecewise $\mathcal{C}^1$.

This approach is more than adequate for every computation that we will do, because our paths of integration always will concatenate finitely many line segments and circular arcs. In this case, assuming without loss of generality that γ is $\mathcal{C}^1$ by working with its pieces one at a time, the complex integral can be treated as the integral of a differential form,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Here the integrand $f(\gamma(t))\gamma'(t)$ is complex-valued, but we simply work with it componentwise. That is, for continuous complex-valued $\varphi(t) = U(t)+iV(t)$ on $[a, b]$, the inevitable definition is $\int_a^b \varphi(t)\,\mathrm{d}t = \int_a^b U(t)\,\mathrm{d}t + i\int_a^b V(t)\,\mathrm{d}t$. In our case, the integrand $\varphi(t) = f(\gamma(t))\gamma'(t)$, with $f = u + iv$ and $\gamma = x + iy$, has components

$U = ux' - vy'$ and $V = vx' + uy'$, with $u = u(x(t), y(t))$, $v = v(x(t), y(t))$, $x' = x'(t)$, and $y' = y'(t)$. Similarly, the appropriate definition here is

$$\int_\gamma f(z)\,|\mathrm{d}z| = \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t,$$

this time integrating the function $\varphi = U + iV$ where $U = u\sqrt{x'^2 + y'^2}$ and $V = v\sqrt{x'^2 + y'^2}$. As a particular case of the second integral, the length of γ is defined as

$$\text{length}(\gamma) = \int_\gamma |\mathrm{d}z| = \int_{t=a}^b |\gamma'(t)|\,\mathrm{d}t.$$

The second approach to complex path integrals is that

γ is assumed to be rectifiable.

Here *rectifiable* means that γ has finite arc length, with arc length defined in a natural way; this will be explained just below. Now the integral definitions are

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^n f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

and

$$\int_\gamma f(z)\,|\mathrm{d}z| = \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^n f(\gamma(c_j))|\gamma(t_j) - \gamma(t_{j-1})|,$$

with the limits being taken over partitions $P = \{t_j\}$ of $[a, b]$, and corresponding samples $S_P = \{c_j\}$,

$$a = t_0 \le c_1 \le t_1 \le c_2 \le t_2 \le \cdots \le t_{n-1} \le c_n \le t_n = b$$

with the t_j distinct, and the mesh of a partition being

$$\text{mesh}(P) = \max_{1\le j\le n} (t_j - t_{j-1}).$$

The fact that these limits exist needs to be established. In this setting, the arc length of γ is

$$\text{length}(\gamma) = \int_\gamma |\mathrm{d}z| = \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|.$$

The assumption that γ is rectifiable is that the set of all sums as in the previous display is bounded above. The values of such sums are the lengths of all polygons inscribed in γ.

If γ is assumed to be $\mathcal{C}^1$, so that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t, \qquad \int_\gamma f(z)\,|\mathrm{d}z| = \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t,$$

and we freely extend to the piecewise $\mathcal{C}^1$ case, some questions present themselves.

- *Invariance: Do these integrals depend on the parametrization of γ?* A short calculation with the chain rule shows that the integral is invariant under order-preserving reparametrization. That is, all that matters is the direction of path-traversal. As for what happens when the direction is reversed, for any path γ, let $-\gamma$ denote the same path but traversed in the opposite direction. Then unsurprisingly,

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z,$$

but on the other hand,

$$\int_{-\gamma} f(z)\,|\mathrm{d}z| = \int_{\gamma} f(z)\,|\mathrm{d}z|.$$

- *Size estimates: How do the two nonnegative real numbers*

$$\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| \qquad \textit{and} \qquad \int_{\gamma} |f(z)|\,|\mathrm{d}z|$$

compare? A slightly clever argument shows that

$$\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| \le \int_{\gamma} |f(z)|\,|\mathrm{d}z|.$$

- *Tradeoffs: What are the advantages of defining complex path integrals in a way that depends on having $\mathcal{C}^1$-paths?* As already mentioned, defining the complex contour integral for piecewise $\mathcal{C}^1$ paths lets us compute every example that we need.

The situation is different when γ is assumed to rectifiable.

- *Invariance.* The integral's invariance under monotonically increasing reparametrizations is essentially wired into its definition. Partitions pass through such reparametrizations, preserving the property of their meshes going to zero.
- *Size estimates.* Similarly, the estimate $|\int_\gamma f(z)\,\mathrm{d}z| \le \int_\gamma |f(z)|\,|\mathrm{d}z|$ is essentially automatic, in consequence of the triangle inequality.
- *Tradeoffs.* The topology theory that underlies complex analysis addresses questions of deforming one path to another through a succession of paths, and those paths are known only to be continuous; it is convenient not to worry whether they are piecewise $\mathcal{C}^1$, although still we have to worry whether they are rectifiable.

We make two more comments before moving on to specifics.

First, the existence of the integrals being discussed here is a substantive question regardless of whether γ is $\mathcal{C}^1$ or only rectifiable. If γ is $\mathcal{C}^1$ then the existence relies on the existence of the integral of a continuous real-valued function over a compact interval. This existence must be invoked until a person is ready to appreciate that it relies on the continuity being uniform, in consequence of the interval being compact. On the other hand, if γ is rectifiable and f is continuous then the existence of the integrals $\int_\gamma f(z)\,\mathrm{d}z$ and $\int_\gamma f(z)\,|\mathrm{d}z|$ doesn't reduce to the real case. Instead a variant existence argument is required, not using the notions of *lower sum* and *upper sum*, because the complex number system is not ordered. This variant argument again boils down to uniform continuity. It will be given below.

Second, maintaining two notions of complex path integrals raises the question of their compatibility. That is, do the *limit of sums* definitions of $\int_\gamma f(z)\,\mathrm{d}z$ and $\int_\gamma f(z)\,|\mathrm{d}z|$ reduce to their *differential form* definitions when the rectifiable path γ is further $\mathcal{C}^1$? They do, and confirming so is good practice with beginning real analysis technique. In particular, the two notions of the length of a $\mathcal{C}^1$ curve γ agree, but one inequality between them is easier to show than the other. It is a recommended conceptual exercise to speculate which inequality should be the easy one, and then to find the nice geometric proof that it is. Here the relevant technical skill is the Cauchy–Schwarz inequality. By contrast, the proof of the harder inequality is more analytic. We will establish both inequalities below.

2. A far-reaching little integral

Before going into generalities, we work a particular path integral that lies at the heart of complex analysis. Our data are as follows.

- r is any positive real number, possibly very large and possibly very small,
- γ_r is the circle of radius r centered at the origin, traversed once counterclockwise,
- n is any integer,
- and $f_n(z) = z^n$. This function is undefined at $z = 0$ if n is negative.

The natural parametrization of γ_r is

$$\gamma_r : [0, 2\pi] \longrightarrow \mathbb{C}, \qquad \gamma_r(t) = re^{it},$$

and so the integral of f_n over γ_r is

$$\begin{aligned}
\int_{\gamma_r} f_n(z)\,\mathrm{d}z &= \int_{t=0}^{2\pi} f_n(\gamma(t))\gamma_r'(t)\,\mathrm{d}t \\
&= \int_{t=0}^{2\pi} (re^{it})^n ire^{it}\,\mathrm{d}t \\
&= \int_{t=0}^{2\pi} r^n e^{int}\, ire^{it}\,\mathrm{d}t \\
&= ir^{n+1}\int_{t=0}^{2\pi} e^{i(n+1)t}\,\mathrm{d}t \\
&= ir^{n+1}\cdot\begin{cases} 2\pi & \text{if } n=-1, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi i & \text{if } n=-1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

That is, the integral

$$\int_{\gamma_r} z^n\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n=-1, \\ 0 & \text{otherwise} \end{cases}$$

is *independent of* r and *nearly independent of* n.

The preceding formula has enormous consequences. For example, naïvely assuming that some function f has a representation in integer powers of z,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

and naïvely assuming that the sum passes through integration over γ_r, it follows that integrating f over $\gamma = \gamma_r$ (for any suitable $r > 0$) picks off the coefficient a_{-1}

of $1/z$ in f and ignores everything else,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = a_{-1}.$$

Making these ideas precise requires some care, and it isn't quite this simple, but things pretty much work out as the calculation here suggests.

3. Invariance of the complex integral

Let Ω be a region in $\mathbb{C}$, let $f:\Omega \longrightarrow \mathbb{C}$ be a continuous function, and consider two rectifiable continuous curves in Ω,

$$\gamma:[a,b]\longrightarrow\Omega, \qquad \tilde\gamma:[c,d]\longrightarrow\Omega.$$

Suppose that $\tilde\gamma$ is an orientation-preserving reparametrization of γ, meaning that there exists a continuous increasing bijection

$$r:[a,b]\longrightarrow[c,d]$$

such that

$$\gamma=\tilde\gamma\circ r.$$

Let P denote any partition of $[a,b]$ with subordinate sample S,

$$P=\{t_0,\dots,t_n\}, \quad S=\{c_1,\dots,c_n\},$$

and similarly for $[c,d]$, with the same number of partition points,

$$\widetilde P=\{\tilde t_0,\dots,\tilde t_n\}, \quad \widetilde S=\{\tilde c_1,\dots,\tilde c_n\},$$

The integrals of f over the two curves are by definition

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{\mathrm{mesh}(P)\to 0}\sum_j f(\gamma(c_j))(\gamma(t_j)-\gamma(t_{j-1})).$$

and

$$\int_{\tilde\gamma} f(z)\,\mathrm{d}z = \lim_{\mathrm{mesh}(\widetilde P)\to 0}\sum_j f(\tilde\gamma(\tilde c_j))(\tilde\gamma(\tilde t_j)-\tilde\gamma(\tilde t_{j-1})).$$

We show that essentially by definition, the two integrals are equal.

The basic idea is that partition-sample pairs for $[c,d]$ and the partition-sample pairs for $[c,d]$ are in bijective correspondence via r and r^{-1},

$$(\widetilde P,\widetilde S)=(r(P),r(S)) \quad\text{and}\quad (P,S)=(r^{-1}(\widetilde P),r^{-1}(\widetilde S)).$$

Thus the sets of Riemann sums for the two integrals are the same. For example, each term $\tilde\gamma(\tilde t_j)$ where $\tilde t_j\in\widetilde P$ is

$$\tilde\gamma(\tilde t_j)=\tilde\gamma(r(t_j))=\gamma(t_j) \quad \text{where } t_j\in P,$$

and similarly each $\gamma(t_j)$ where $t_j\in P$ is $\tilde\gamma(\tilde t_j)$ where $\tilde t_j\in\widetilde P$.

The nice little technical point here is that the inverse bijection r^{-1} is also continuous. To show this, it suffices to show that r takes closed sets to closed sets. But in this context, closed and compact mean the same thing, and indeed r takes compact sets to compact sets because the continuous image of a compact set is compact.

Now, because r and r^{-1} are uniformly continuous, it follows that if $\{(P_m,S_m)\}$ and $\{(\widetilde P_m,\widetilde S_m)\}$ are partition-sample sequences related to each other via r and r^{-1} then

$$\lim\{\mathrm{mesh}(P_m)\}=0 \iff \lim\{\mathrm{mesh}(\widetilde P_m)\}=0.$$

These considerations show that the two integrals are equal,

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \lim_{\mathrm{mesh}(P)\to 0}\sum_j f(\gamma(c_j))(\gamma(t_j)-\gamma(t_{j-1}))\\ &= \lim_{\mathrm{mesh}(\tilde{P})\to 0}\sum_j f(\tilde{\gamma}(\tilde{c}_j))(\tilde{\gamma}(\tilde{t}_j)-\tilde{\gamma}(\tilde{t}_{j-1}))\\ &= \int_{\tilde{\gamma}} f(z)\,\mathrm{d}z.\end{aligned}$$

If the curves γ and $\tilde{\gamma}$ are $\mathcal{C}^1$ then the invariance result is automatic, using the chain rule for the third equality to follow, and then the change of variable theorem from one variable calculus for the fifth, the latter theorem used twice because of the complex integrand,

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b (f\circ\gamma)\cdot\gamma'\\ &= \int_a^b (f\circ\tilde{\gamma}\circ r)\cdot(\tilde{\gamma}\circ r)'\\ &= \int_a^b (f\circ\tilde{\gamma}\circ r)\cdot(\tilde{\gamma}'\circ r)\cdot r'\\ &= \int_a^b (((f\circ\tilde{\gamma})\cdot\tilde{\gamma}')\circ r)\cdot r'\\ &= \int_c^d (f\circ\tilde{\gamma})\cdot\tilde{\gamma}'\\ &= \int_{\tilde{\gamma}} f(z)\,\mathrm{d}z.\end{aligned}$$

But this immediate argument is subsumed by the argument for rectifiable continuous paths.

4. The basic complex integral estimate

We show the following result.

> *Let Ω be a region, let $f:\Omega\longrightarrow\mathbb{C}$ be a continuous function, and let $\gamma:[a,b]\longrightarrow\Omega$ be a rectifiable path. Then*
>
> $$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \int_\gamma |f(z)|\,|\mathrm{d}z|.$$

To see this, recall that the integral is the limit of Riemann sums, and compute (using the fact that the absolute value function is continuous and using the triangle inequality) that

$$\begin{aligned}
\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\lim_{\mathrm{mesh}(P)\to 0}\sum_{j=1}^n f(\gamma(c_j))(\gamma(t_j)-\gamma(t_{j-1}))\right| \\
&= \lim_{\mathrm{mesh}(P)\to 0}\left|\sum_{j=1}^n f(\gamma(c_j))(\gamma(t_j)-\gamma(t_{j-1}))\right| \\
&\le \lim_{\mathrm{mesh}(P)\to 0}\sum_{j=1}^n |f(\gamma(c_j))|\,|(\gamma(t_j)-\gamma(t_{j-1}))| \\
&= \int_\gamma |f(z)|\,|\mathrm{d}z|.
\end{aligned}$$

Truly, this is all there is to it.

Here also is the standard argument (in our text, for example) given the stronger hypothesis that γ is a $\mathcal{C}^1$-path. This argument is required if one has defined the complex integral *only* for (piecewise) $\mathcal{C}^1$-paths, by parametrization, rather than for rectifiable curves, by Riemann sums. It is striking, at least to the author of this note, that the following proof, despite having stronger hypotheses than the previous one, is more complicated. In this instance, avoiding the Riemann sum definition of the integral makes things harder rather than easier.

The comparable result in the *real* setting,

$$\left|\int_{t=a}^b \varphi(t)\,\mathrm{d}t\right| \le \int_{t=a}^b |\varphi(t)|\,\mathrm{d}t, \qquad (\varphi : [a,b] \longrightarrow \mathbb{R} \text{ integrable}),$$

is easy: simply integrate the relation $-|\varphi(t)| \le \varphi(t) \le |\varphi(t)|$ over the interval $[a,b]$. And the complex case should be essentially no harder. However, reducing the complex case to the real case poses two obstacles: first, $f(z)$ takes complex values, and second, $\mathrm{d}z$ is a complex differential. The following argument addresses these issues one at a time, first reducing the problem to the case of a complex-valued function to that of a real-valued one, assuming that the differential is real-valued, and then using the definitions $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$, $|\mathrm{d}z| = |\gamma'(t)|\,\mathrm{d}t$ to reduce the case of a complex differential to that of a real one.

So, begin by considering a continuous complex-valued function on a real interval,

$$\varphi : [a,b] \longrightarrow \mathbb{C}.$$

The claim is that

$$\left|\int_{t=a}^b \varphi(t)\,\mathrm{d}t\right| \le \int_{t=a}^b |\varphi(t)|\,\mathrm{d}t.$$

If $\int_{t=a}^b \varphi(t)\,\mathrm{d}t = 0$ then the claim holds, so we may take $\int_{t=a}^b \varphi(t)\,\mathrm{d}t = re^{i\theta}$, $r > 0$. Thus

$$\begin{aligned}
\left|\int_{t=a}^b \varphi(t)\,\mathrm{d}t\right| &= r = e^{-i\theta}re^{i\theta} = e^{-i\theta}\int_{t=a}^b \varphi(t)\,\mathrm{d}t = \int_{t=a}^b e^{-i\theta}\varphi(t)\,\mathrm{d}t \\
&= \int_{t=a}^b (\mathrm{Re}(e^{-i\theta}\varphi(t)) + i\mathrm{Im}(e^{-i\theta}\varphi(t)))\,\mathrm{d}t.
\end{aligned}$$

But the integral is real, so its imaginary part is zero, leaving us in a position to quote the inequality from the real case and then quote the fact that the size of the real component is at most the size of the complex number,

$$\begin{aligned}\left|\int_{t=a}^{b} \varphi(t)\,\mathrm{d}t\right| = \int_{t=a}^{b} \mathrm{Re}(e^{-i\theta}\varphi(t))\,\mathrm{d}t &\leq \int_{t=a}^{b} |\mathrm{Re}(e^{-i\theta}\varphi(t))|\,\mathrm{d}t \\ &\leq \int_{t=a}^{b} |e^{-i\theta}\varphi(t)|\,\mathrm{d}t = \int_{t=a}^{b} |\varphi(t)|\,\mathrm{d}t.\end{aligned}$$

Now the general result follows. Let $\varphi = (f \circ \gamma) \cdot \gamma'$. Then

$$\begin{aligned}\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| = \left|\int_{t=a}^{b} \varphi(t)\,\mathrm{d}t\right| &\leq \int_{t=a}^{b} |\varphi(t)|\,\mathrm{d}t \\ &= \int_{t=a}^{b} |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t = \int_{\gamma} |f(z)|\,|\mathrm{d}z|.\end{aligned}$$

5. Compatibility

Let $\Omega \subset \mathbb{C}$ be a region, let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function, and let $\gamma : [a,b] \longrightarrow \Omega$ be a $\mathcal{C}^1$ path. Recall our two definitions of the integral of f over γ, the first using the derivative of γ and then a definition of the Riemann integral over a real interval,

$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{a}^{b} f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \lim_{\mathrm{mesh}(P)\to 0} \sum_{j} f(\gamma(c_j))\gamma'(c_j)(t_j - t_{j-1})$$

and the second making no reference to the derivative,

$$\int_{\gamma} f(z)\,\mathrm{d}z = \lim_{\mathrm{mesh}(P)\to 0} \sum_{j} f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})).$$

Note that the Riemann integral definition here is not the version that uses lower and upper sums, but instead defines the integral as a common limit over all sequences of partitions whose meshes go to zero. The next section will show that this integral exists, in consequence of showing that a complex integral exists with no assumption of differentiability. Here, taking the *mesh* definition of the real Riemann integral for granted, we sketch the argument that the two definitions just given for the complex integral are compatible.

Part of the summand in the first definition is, letting $\gamma = (x, y)$,

$$\gamma'(c_j)(t_j - t_{j-1}) = \big(x'(c_j) + iy'(c_j)\big)\,(t_j - t_{j-1}),$$

while two applications of the Mean Value Theorem show that part of the summand in the second definition is

$$\gamma(t_j) - \gamma(t_{j-1}) = \big(x'(d_j) + iy'(e_j)\big)\,(t_j - t_{j-1}), \quad \text{for some } d_j, e_j \in (t_{j-1}, t_j).$$

Thus the difference of the summands is

$$f(\gamma(c_j))\,\big(x'(c_j) - x'(d_j) + i(y'(c_j) - y'(e_j))\big)\,(t_j - t_{j-1}).$$

Because f is continuous and the trace of γ is compact (it is the continuous image of the compact set $[a,b]$), f is bounded on the trace of γ. Also, because γ is $\mathcal{C}^1$, its component function derivatives x' and y' are continuous on $[a,b]$, and because

$[a,b]$ is compact, they are uniformly continuous there. Therefore, given $\varepsilon > 0$, if the partition P is fine enough then for each j,

$$|f(\gamma(c_j))\left(x'(c_j) - x'(d_j) + i(y'(c_j) - y'(e_j))\right)(t_j - t_{j-1})| < \frac{\varepsilon(t_j - t_{j-1})}{b-a}.$$

This makes the two sums within ε of each other. Thus the integrals are equal.

6. Compatibility of arc length

Let $\Omega \subset \mathbb{C}$ be a region, and let $\gamma : [a,b] \longrightarrow \Omega$ be a $\mathcal{C}^1$ path. Recall our two definitions of the length of γ, the first using the derivative of γ,

$$\operatorname{length}(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t,$$

and the second being the supremum of inscribed polygonal path-lengths, making no reference to the derivative,

$$\operatorname{length}(\gamma) = \sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})|.$$

We show that the definitions are compatible.

It is not hard to establish that the integral of $|\gamma'|$ is at least the length of any inscribed polygonal path, because these lengths grow under refinement and the integral is conceptually their limit. Indeed, take a partition of $[a,b]$,

$$P = \{t_0, t_1, \ldots, t_n\},$$

and assume that no consecutive pair of division points t_{j-1} and t_j have the same image under γ. (If $\gamma(t_{j-1}) = \gamma(t_j)$ then the pair contributes nothing to the length of the polygonal path, and so we may drop its second point from the overall calculation.) Fix any $j \in \{1, \ldots, n\}$. Consider the unit vector in the direction between the jth pair of consecutive polygon points,

$$\hat{v} = (\gamma(t_j) - \gamma(t_{j-1}))/|\gamma(t_j) - \gamma(t_{j-1})|.$$

The trivial estimate $\hat{v} \cdot \gamma'(t) \le |\hat{v} \cdot \gamma'(t)|$, then the Cauchy–Schwarz inequality $|\hat{v} \cdot \gamma'(t)| \le |\hat{v}|\,|\gamma'(t)| = |\gamma'(t)|$ give the inequality in the calculation

$$\begin{aligned} |\gamma(t_j) - \gamma(t_{j-1})| &= \hat{v} \cdot (\gamma(t_j) - \gamma(t_{j-1})) \\ &= \hat{v} \cdot \int_{t_{j-1}}^{t_j} \gamma'(t)\,\mathrm{d}t = \int_{t_{j-1}}^{t_j} \hat{v} \cdot \gamma'(t)\,\mathrm{d}t \le \int_{t_{j-1}}^{t_j} |\gamma'(t)|\,\mathrm{d}t. \end{aligned}$$

That is, the jth inscribed polygonal segment length is at most the jth piece of the integral. Sum over j to get the inequality

$$\sum_j |\gamma(t_j) - \gamma(t_{j-1})| \le \int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

This holds for any partition P, and the right side is independent of P. It follows that

$$\sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})| \le \int_a^b |\gamma'(t)|\,\mathrm{d}t. \tag{1}$$

The opposite inequality is more delicate. The idea is to get polygonal path-lengths as close to the integral of $|\gamma'|$ as desired. The argument to follow can prove both inequalities, but the easier direction deserved its correspondingly smoother proof.

The derivatives $x'(t)$ and $y'(t)$ of the component functions of $\gamma(t)$ are continuous, and their domain $[a,b]$ is compact, so they are uniformly continuous on their domain. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ so that

$$\left\{\begin{matrix} t, \tilde{t} \in [a,b], \\ |\tilde{t} - t| < \delta \end{matrix}\right\} \implies \max\{|x'(\tilde{t}) - x'(t)|, |y'(\tilde{t}) - y'(t)|\} < \frac{\varepsilon}{4(b-a)}.$$

So if P partitions $[a,b]$ more finely than δ then for any $j \in \{1,\dots,n\}$ and for any $s_j, \tilde{s}_j \in (t_{j-1}, t_j)$, the reverse triangle inequality gives

$$\begin{aligned} \big|\big(x'(s_j), y'(\tilde{s}_j)\big)\big| &= \big|\big(x'(t_j), y'(t_j)\big) \;-\; \big(x'(t_j) - x'(s_j), y'(t_j) - y'(\tilde{s}_j)\big)\big| \\ &\geq \big|\big(x'(t_j), y'(t_j)\big)\big| \;-\; \big|\big(x'(t_j) - x'(s_j), y'(t_j) - y'(\tilde{s}_j)\big)\big| \\ &> \big|\big(x'(t_j), y'(t_j)\big)\big| - \frac{\varepsilon}{2(b-a)} \\ &= |\gamma'(t_j)| - \frac{\varepsilon}{2(b-a)}. \end{aligned}$$

Now compute for any j, using the Mean Value Theorem twice at the first step and using the previous calculation at the last step, that

$$\begin{aligned} |\gamma(t_j) - \gamma(t_{j-1})| &= \big|\big(x'(s_j), y'(\tilde{s}_j)\big)\,(t_j - t_{j-1})\big| \quad \text{for some } s_j, \tilde{s}_j \in (t_{j-1}, t_j) \\ &= \big|\big(x'(s_j), y'(\tilde{s}_j)\big)\big|\,(t_j - t_{j-1}) \\ &> |\gamma'(t_j)|\,(t_j - t_{j-1}) - \frac{\varepsilon}{2(b-a)}(t_j - t_{j-1}). \end{aligned}$$

Sum over j to get

$$\sum_j |\gamma(t_j) - \gamma(t_{j-1})| > \sum_j |\gamma'(t_j)|(t_j - t_{j-1}) - \frac{\varepsilon}{2}.$$

But if the partition is fine enough, then by the definition of the Riemann integral we also have

$$\sum_j |\gamma'(t_j)|(t_j - t_{j-1}) > \int_a^b |\gamma'(t)|\,\mathrm{d}t - \frac{\varepsilon}{2}.$$

Combining the last two displays gives

$$\sum_j |\gamma(t_j) - \gamma(t_{j-1})| > \int_a^b |\gamma'(t)|\,\mathrm{d}t - \varepsilon,$$

and it follows trivially that

$$\sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})| > \int_a^b |\gamma'(t)|\,\mathrm{d}t - \varepsilon.$$

Because this holds for all $\varepsilon > 0$,

$$\sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})| \geq \int_a^b |\gamma'(t)|\,\mathrm{d}t. \tag{2}$$

Equations (1) and (2) together give the result.

7. Existence of the integral

The outstanding issue is that for a region $\Omega \subset \mathbb{C}$, a continuous function $f : \Omega \longrightarrow \mathbb{C}$, and a rectifiable continuous path $\gamma : [a, b] \longrightarrow \Omega$, the integral $\int_\gamma f(z)\,\mathrm{d}z$ exists. Recall that a *partition* of $[a, b]$ is a set

$$P = \{t_0, t_1, \dots, t_n\}$$

where

$$a = t_0 < t_1 < \cdots < t_n = b.$$

The number $n \geq 0$ can vary from partition to partition. The *mesh of* P is the maximum of the lengths of the subintervals determined by P,

$$\operatorname{mesh}(P) = \max_j \{t_j - t_{j-1}\}.$$

Refining the partition P cannot increase its mesh. A *sample subordinate to* P is a set

$$S_P = \{c_1, \dots, c_n\}$$

where

$$t_0 \leq c_1 \leq t_1 \leq \cdots \leq t_{n-1} \leq c_n \leq t_n.$$

As above, the definition of the integral of f over γ is a very general limit, if it exists,

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{\operatorname{mesh}(P)\to 0} \sum_{j=1}^{n} f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})).$$

The sum on the right is denoted $\Sigma(P, S_P)$.

Existence Theorem. *Let $\Omega \subset \mathbb{C}$ be a region, let $f : \Omega \longrightarrow \mathbb{C}$ be continuous, and let $\gamma : [a, b] \longrightarrow \Omega$ be continuous and rectifiable. Then the integral $\int_\gamma f(z)\,\mathrm{d}z$ exists.*

Proof. Take a sequence of partitions $\{P_1, P_2, \dots, P_N, \dots\}$ such that

$$\lim_N \operatorname{mesh}(P_N) = 0.$$

We need to show that the limit

$$\lim_N \Sigma(P_N, S_{P_N})$$

exists independently of the particular sequence $\{P_N\}$ of partitions used and independently of the sample S_{P_N} chosen for each partition P_N.

Because $[a, b]$ is compact and γ is continuous, the trace of γ,

$$\hat{\gamma} = \{\gamma(t) : t \in [a, b]\}$$

is compact. Consequently f is uniformly continuous on $\hat{\gamma}$. Let $\varepsilon > 0$ be given, and let L denote the length of γ. Then there exists some $\delta > 0$ such that for all $z, z' \in \hat{\gamma}$,

$$|z - z'| < \delta \implies |f(z) - f(z')| < \frac{\varepsilon}{2L}.$$

Also because $[a, b]$ is compact, γ is uniformly continuous on $[a, b]$, so there exists some $\rho > 0$ such that for all $t, t' \in [a, b]$,

$$|t - t'| < \rho \implies |\gamma(t) - \gamma(t')| < \delta.$$

Consequently, for all $t, t' \in [a, b]$,

$$|t - t'| < \rho \implies |f(\gamma(t)) - f(\gamma(t'))| < \frac{\varepsilon}{2L}.$$

Take an index $\widetilde{N}$ large enough that

$$\operatorname{mesh}(P_N) < \rho \quad \text{for any } N > \widetilde{N}.$$

Claim: For any $N > \widetilde{N}$, for any refinement Q of P_N, and for any sample S_Q,

$$|\Sigma(P_N, S_{P_N}) - \Sigma(Q, S_Q)| < \frac{\varepsilon}{2}.$$

To prove this, note that corresponding to each jth term of $\Sigma(P_N, S_N)$,

$$f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})),$$

we have a jth *sum* of terms in $\Sigma(Q, S_Q)$,

$$\sum_{k=1}^{m} f(\gamma(c'_k))(\gamma(t'_k) - \gamma(t'_{k-1})),$$

with $t'_0 = t_{j-1}$ and $t'_m = t_j$. The jth term $f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1}))$ of $\Sigma(P_N, S_{P_N})$ rewrites as a telescoping sum,

$$\sum_{k=1}^{m} f(\gamma(c_j))(\gamma(t'_k) - \gamma(t'_{k-1})).$$

Thus,

$$\begin{aligned}
&|j\text{th term in } \Sigma(P_N, S_{P_N}) - j\text{th sum in } \Sigma(Q, S_Q)| \\
&\qquad = |\sum_{k=1}^{m} (f(\gamma(c_j)) - f(\gamma(c'_k)))(\gamma(t'_k) - \gamma(t'_{k-1}))| \\
&\qquad \le \sum_{k=1}^{m} |f(\gamma(c_j)) - f(\gamma(c'_k))||\gamma(t'_k) - \gamma(t_{k-1})| \\
&\qquad < \frac{\varepsilon}{2L} \operatorname{length}(\gamma|_{[t_{j-1}, t_j]}),
\end{aligned}$$

where we have used the fact that $\operatorname{mesh}(P_N) < \rho$ at the last step. Now sum over j to get

$$|\Sigma(P_N, S_{P_N}) - \Sigma(Q, S_Q)| < \frac{\varepsilon}{2L} \operatorname{length}(\gamma) = \frac{\varepsilon}{2}.$$

This proves the claim.

Now, the claim proves that for any N and M greater than $\tilde{N}$,

$$|\Sigma(P_N, S_{P_N}) - \Sigma(P_M, S_{P_M})| < \varepsilon.$$

To see this, let Q be the common refinement of P_N and P_M, meaning their union, and let S_Q be any corresponding sample. Then by the claim and the triangle inequality,

$$\begin{aligned}
&|\Sigma(P_N, S_{P_N}) - \Sigma(P_M, S_{P_M})| \\
&\quad \le |\Sigma(P_N, S_{P_N}) - \Sigma(Q, S_Q)| + |\Sigma(Q, S_Q) - \Sigma(P_M, S_{P_M})| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This pretty much proves the theorem in turn. We now know that the complex sequence

$$\{\Sigma(P_N, S_{P_N})\}$$

is a Cauchy sequence. By the completeness of $\mathbb{C}$, the sequence converges to some complex number

$$\lim_N \{\Sigma(P_N, S_{P_N})\}.$$

What remains to be shown is that this limit is independent of which sequence of partitions P_N we chose and of which subordinate sample S_{P_N} we chose for each P_N. But if $\{P'_N\}$ is another such sequence of partitions, or the same sequence but with different samples, then the complex sequence

$$\{\Sigma(P'_N, S_{P'_N})\}$$

also converges, to some complex number

$$\lim_N \{\Sigma(P'_N, S_{P'_N})\}.$$

The blended sequence

$$\{\Sigma(P_1, S_{P_1}), \Sigma(P'_1, S_{P'_1}), \Sigma(P_2, S_{P_2}), \Sigma(P'_2, S_{P'_2}), \dots\}$$

again converges because the meshes of the blended sequence of partitions go to 0, and its limit must be both of the previous limits because each is the limit of a subsequence. Thus the limit has the desired independence properties, making it a suitable definition of the integral. $\square$