MATH 311: COMPLEX ANALYSIS — MAPPINGS LECTURE

1. The complex exponential

The exponential function

$$\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$$

is defined to be

$$\exp(x+iy) = e^x e^{iy} \quad \text{where } e^{iy} = \cos y + i \sin y.$$

It is natural to think of the inputs to the exponential map in cartesian coordinates, and of the outputs in polar coordinates. Thus,

$$|\exp(x+iy)| = e^x \quad \text{and} \quad \arg(\exp(x+iy)) = y + 2\pi\mathbb{Z}.$$

To tidy up the notation, define

$$e^z = \exp(z).$$

This definition expands in cartesian coordinates to

$$e^{x+iy} = e^x e^{iy}.$$

We already know from a little trigonometry that $e^{i(y+y')} = e^{iy}e^{iy'}$, as follows:

$$\begin{aligned} e^{i(y+y')} &= \cos(y+y') + i\sin(y+y'), \\ e^{iy}e^{iy'} &= \big(\cos(y) + i\sin(y)\big)\big(\cos(y') + i\sin(y')\big) \\ &= \big(\cos(y)\cos(y') - \sin(y)\sin(y')\big) + i\big(\sin(y)\cos(y') + \cos(y)\sin(y')\big), \end{aligned}$$

and so the relation $e^{i(y+y')} = e^{iy}e^{iy'}$ is equivalent to the trigonometry addition law formulas $\cos(y+y') = \cos(y)\cos(y') - \sin(y)\sin(y')$ and $\sin(y+y') = \sin(y)\cos(y') + \cos(y)\sin(y')$. Consequently, also

$$e^{z+z'} = e^z e^{z'}.$$

That is, the exponential map is a homomorphism from the additive group $(\mathbb{C}, +)$ to the multiplicative group $(\mathbb{C} - \{0\}, \cdot)$.

The exponential map has kernel $2\pi i\mathbb{Z}$. Since it is a homomorphism, it gives rise to an isomorphism that can also be denoted *exp*,

$$\exp : (\mathbb{C}/2\pi i\mathbb{Z}, +) \xrightarrow{\sim} (\mathbb{C} - \{0\}, \cdot).$$

Visually, we can imagine the complex exponential map as rolling the plane into a tube and then looking down the tube lengthwise. But this isn't fully accurate since the exponential radial magnification is inconsistent with the laws of perspective.

2. The complex logarithm

The natural definition of the complex logarithm is

$$\log(z) = w \iff \exp(w) = z, \qquad z \neq 0.$$

However, since *exp* is many-to-one, this makes the complex logarithm multiple valued. Since *exp* is the cartesian-to-polar coordinate map but with exponential radial scaling, the formula for the complex logarithm must be

$$\log(z) = \ln|z| + i \arg z.$$

Indeed this is multiple-valued, since *arg* is defined only modulo $2\pi\mathbb{Z}$. To make it single-valued, make its range be the same quotient as above,

$$\log : \mathbb{C} - \{0\} \xrightarrow{\sim} \mathbb{C}/2\pi i\mathbb{Z}.$$

Alternatively, we can take branches of *log*, or we can define *log* on a Riemann surface rather than on the punctured plane. The formula

$$\log(zz') = \log(z) + \log(z')$$

does *not* hold in general for branches of log, but it does hold when the logarithm maps to a quotient or is defined on a suitable Riemann surface.

3. Complex powers

The general power map is

$$z^a = \exp(a \log z), \quad a, z \in \mathbb{C},\ z \neq 0.$$

This formula expands to

$$z^a = \exp(a \ln|z| + ia \arg(z)).$$

3.1. **Complex base, real exponent.** When the exponent a is a real number,

$$z^a = e^{a \ln|z|} e^{ia \arg(z)} = |z|^a e^{ia \arg(z)}, \quad a \in \mathbb{R},$$

i.e.,

$$\boxed{|z^a| = |z|^a \quad \text{and} \quad \arg(z^a) = a \arg(z), \qquad a \in \mathbb{R}.}$$

Especially, if $a = n \in \mathbb{Z}$ then we recover the polar interpretation of powers of complex numbers,

$$|z^n| = |z|^n \quad \text{and} \quad \arg(z^n) = n \arg(z).$$

For example, the case $n = 2$ recovers the result that squaring a complex number squares its modulus and doubles its argument. Also, the case $n = -1$ is worth internalizing: the complex inverse function inverts modulus and negates angle,

$$|z^{-1}| = |z|^{-1} \quad \text{and} \quad \arg(z^{-1}) = -\arg(z).$$

The net effect is to preserve two-dimensional orientation. In particular, the inverse is the conjugate on the unit circle,

$$z^{-1} = \overline{z}, \quad |z| = 1.$$

Also, if $a = 1/n$ where $n \in \mathbb{Z}^+$ then we recover the polar interpretation of roots of complex numbers,

$$|z^{1/n}| = |z|^{1/n} \quad \text{and} \quad \arg(z^{1/n}) = (1/n) \arg(z).$$

Note that here,

$$\text{if} \quad \arg(z) = \theta_0 + 2\pi\mathbb{Z} \quad \text{then} \quad (1/n)\arg(z) = \theta_0/n + (2\pi/n)\mathbb{Z}.$$

This says that the nth root is n-valued. Alternatively, the nth root can be defined on a Riemann surface, a quotient of the Riemann surface for the logarithm.

3.2. **Positive real base, complex exponent.** Switching the roles of a and z, suppose now that the base a is a positive real number and that the exponent z is an arbitrary complex number. The definition of the power map becomes

$$a^z = \exp(z \log a), \quad a > 0.$$

But here it is convenient to restrict the logarithm to the principal branch, or any other branch such that $\log a = \ln a$. The definition becomes

$$a^z = \exp(z \ln a) = e^{x \ln a + iy \ln a} = a^x e^{iy \ln a},$$

and in particular,

$$|a^z| = a^{\mathrm{Re}(z)}, \quad a > 0,\ z \in \mathbb{C}.$$

This estimate will be useful later in the course. For example, it shows that the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C},$$

converges absolutely on the right half-plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$, and in fact the absolute convergence is uniform on compacta.

4. The function $(z + z^{-1})/2$

Consider the function

$$f : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{z + z^{-1}}{2}.$$

Because $f(z^{-1}) = f(z)$, we may analyze the function on the punctured closed disk, $\overline{D}^{\circ} = \{z : 0 < |z| \le 1\}$. Since $z^{-1} = \overline{z}$ on the boundary circle, f on the boundary circle picks off the real part of z, projecting the upper and lower halves of the boundary circle to the segment $[-1, 1]$. Inside the boundary circle, the term z^{-1} of f is larger in magnitude than the term z, so in anticipation of studying f inside the disk, we should imagine the top half of the boundary circle being projected to the lower side of the segment, and the bottom half of the circle to the top side of the segment.

To study f further, write its input values z in polar coordinates, and compute

$$f(re^{i\theta}) = \frac{re^{i\theta} + r^{-1}e^{-i\theta}}{2} = \frac{r + r^{-1}}{2}\cos\theta + i\frac{r - r^{-1}}{2}\sin\theta.$$

Let $r = e^t$ where $-\infty < t \le 0$. Then

$$f(re^{i\theta}) = \cosh t \cos\theta + i \sinh t \sin\theta \overset{\text{call}}{=} u + iv.$$

Concentric circles in the punctured disk are traversed by fixing t and letting θ vary. We have

$$\left(\frac{u}{\cosh t}\right)^2 + \left(\frac{v}{\sinh t}\right)^2 = \cos^2\theta + \sin^2\theta = 1,$$

showing that concentric circles map to ellipses. In the unit disk, the dominant term of f is $z^{-1}/2$. Thus, as r shrinks from 1 down toward 0, the ellipses expand about

$[-1, 1]$, and if the circles are traversed counterclockwise, the ellipses are traversed clockwise. This reversal of two linear dimensions preserves planar orientation.

On the other hand, radii in the punctured disk are traversed by fixing θ and letting t vary. We have

$$\left(\frac{u}{\cos\theta}\right)^2 - \left(\frac{v}{\sin\theta}\right)^2 = \cosh^2 t - \sinh^2 t = 1,$$

showing that radii map to hyperbolas. As θ runs counterclockwise from 0 to 2π, the hyperbola segments progress clockwise, but again this reversal is balanced by the fact that as z moves away from the origin along the radii, $f(z)$ moves toward $[-1, 1]$ along the hyperbolas. Again, we see that the map preserves planar orientation.

Symbolically, we invert the map as follows. Let

$$w = \frac{z + z^{-1}}{2}.$$

Then

$$z^2 - 2wz + 1 = 0,$$

and so by the quadratic formula,

$$z = w + \sqrt{w^2 - 1}.$$

The square root is double-valued (reflecting the fact that f is 2-to-1), and so we need to decide how to proceed.

One option is to define f on a quotient space of its original domain, obtained by identifying reciprocal pairs $\{z, z^{-1}\}$ as a single point. This quotient is topologically a punctured sphere. If we define $1/\infty = 0$ then f is defined at 0 and at ∞, and the quotient becomes a complete sphere. This option makes the environment smaller.

A second option is to take a branch of $\sqrt{w^2 - 1} = \sqrt{w - 1}\sqrt{w + 1}$. The first square root changes its sign when a small circle about $w = 1$ is traversed, and similarly for the second square root and $w = -1$. Taking $[-1, 1]$ as the branch cut forces any circle around one endpoint to go around the other as well, giving us a choice between two well defined products of the square roots. By convention, we let $\sqrt{w^2 - 1}$ denote the choice that takes positive values for $w > 1$. This option looks at most but not all of the environment.

A third option is to create a Riemann surface for the inverse function. View the exterior of the unit disk as being taken by f to a second copy of the plane, with the two copies suitably glued together at the two sides of the slit $[-1, 1]$. Again adding in 0 and ∞, this makes the Riemann surface for the inverse function a sphere. This option makes the environment bigger.

Life is not all spheres. Let $a, b, c, d \in \mathbb{C}$ be distinct complex numbers, and consider the double-valued function

$$h(w) = \sqrt{(w - a)(w - b)(w - c)(w - d)}.$$

The Riemann surface for h is a torus.