JACOBI'S BASIC THETA FUNCTION

Define

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z + \pi i n^2 \tau}, \quad \text{where} \quad \left\{ \begin{array}{l} z = x+iy \in \mathbb{C} \\ \tau = \sigma + it \in \mathcal{H} \end{array} \right\}.$$

Our main use of the theta function will be that its restriction to $z = 0$ is a modular form in τ, and the Euler–Riemann zeta function is essentially the Mellin transform of the modular form: this is one way that Riemann established the analytic continuation and functional equation of zeta. But this writeup will briefly describe how the theta function solves the heat equation. Theta functions in general are powerfully versatile. David Mumford's *Tata Lectures on Theta* volumes are an excellent source of reading on this topic.

The magnitude of the $\theta(x+iy, \sigma+it)$ summand is

$$|e^{2\pi i n(x+iy)+\pi i n^2(\sigma+it)}| = e^{-2\pi n y - \pi n^2 t} = o(e^{-\pi|n|t}) \quad \text{as } |n| \to \infty.$$

Thus the series converges rapidly for any $t > 0$, thanks to the summand-component $e^{\pi i n^2 \tau}$.

Clearly $\theta(z,\tau)$ is $\mathbb{Z}$-periodic in z. (Its periodicity in τ, both stronger and more subtle than in z, will be discussed in a separate writeup.)

Restrict the theta function to $z = x \in \mathbb{R}$ and $\tau = it$ with $t > 0$, and give the restriction $\theta(x, it)$—which takes real values—its own name,

$$u_\theta : \mathbb{R} \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}, \qquad u_\theta(x,t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x - \pi n^2 t}.$$

The idea is that x is a position variable and t a time variable, and the restriction visibly satisfies the heat equation,

$$\begin{aligned} \frac{\partial^2 u_\theta}{\partial x^2}(x,t) &= \sum_{n\in\mathbb{Z}} (-4\pi^2 n^2) e^{2\pi i n x - \pi n^2 t} \\ &= 4\pi \sum_{n\in\mathbb{Z}} (-\pi n^2) e^{2\pi i n x - \pi n^2 t} = 4\pi \frac{\partial u_\theta}{\partial t}(x,t). \end{aligned}$$

Furthermore, the limiting initial time-value of $u(x,t)$ is the $\mathbb{Z}$-periodicized Dirac delta function in the position variable,

$$\lim_{t\to 0^+} u_\theta(x,t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x} = \left\{ \begin{array}{ll} \infty & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{array} \right\} = \delta^{\text{per}}(x).$$

Thus we have a periodic solution of the heat equation whose limiting initial condition is particularly handy.

The general heat equation is now easy to solve. Consider an arbitrary periodic initial time-value condition u_o. Let v_o be its restriction to $[0,1)$ extended by zero to $\mathbb{R}$, so that u_o is its $\mathbb{Z}$-periodicization,

$$u_o(x) = \sum_{n\in\mathbb{Z}} v_o(x+n).$$

Consider the convolution of u_θ and v_o,

$$u = u_\theta * v_o : \mathbb{R}\times\mathbb{R}_{>0} \longrightarrow \mathbb{R}, \qquad u(x,t) = \int_{\mathbb{R}} u_\theta(x-\widetilde{x},t)v_o(\widetilde{x})\,d\widetilde{x}.$$

Differentiation under the integral sign shows that u again satisfies the heat equation,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,t) &= \int_{\mathbb{R}} \frac{\partial^2 u_\theta}{\partial x^2}(x-\widetilde{x},t)v_o(\widetilde{x})\,d\widetilde{x}\\ &= 4\pi\int_{\mathbb{R}} \frac{\partial u_\theta}{\partial t}(x-\widetilde{x},t)v_o(\widetilde{x})\,d\widetilde{x} = 4\pi\frac{\partial u}{\partial t}(x,t).\end{aligned}$$

Furthermore, the limiting initial-time value of u is

$$\begin{aligned}\lim_{t\to 0^+} u(x,t) &= \int_{\mathbb{R}} \lim_{t\to 0^+} u_\theta(x-\widetilde{x},t)v_o(\widetilde{x})\,d\widetilde{x}\\ &= \int_{\mathbb{R}} \delta^{\text{per}}(x-\widetilde{x})v_o(\widetilde{x})\,d\widetilde{x} = \sum_{n\in\mathbb{Z}} v_o(x+n) = u_o(x).\end{aligned}$$

Thus, modulo many informalities, the theta function solves the periodic heat equation.