THE COMPLEX FUNDAMENTAL THEOREM OF CALCULUS

1. The rectifiable curve case

Let

- Ω be a region in $\mathbb{C}$,
- $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function such that $f = F'$ for some F,
- $\gamma : [a, b] \longrightarrow \Omega$ be a continuous and rectifiable curve.

We show that

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Because γ is assumed only to be rectifiable rather than $\mathcal{C}^1$, this result does not simply reduce to the real case.

Let $\varepsilon > 0$ be given.

There exists $\mu > 0$ such that for any partition $P = \{t_0, t_1, \dots, t_n\}$ of $[a, b]$ having mesh less than μ, and for any subordinate sample $S_P = \{c_1, \dots, c_n\}$, if we set $\zeta_j = \gamma(t_j)$ for $j = 0, \dots, n$ and $z_j = \gamma(c_j)$ for $j = 1, \dots, n$, then

$$\left| \sum_j f(z_j)(\zeta_j - \zeta_{j-1}) - \int_\gamma f(z)\,\mathrm{d}z \right| < \frac{\varepsilon}{2}\,. \tag{1}$$

For each z on γ, because $f(z) = F'(z)$ we have, using a variable ζ,

$$F(\zeta) - F(z) - f(z)(\zeta - z) = \mathrm{o}(\zeta - z),$$

which is to say that for some positive radius $r(z) > 0$,

$$|F(\zeta) - F(z) - f(z)(\zeta - z)| < \frac{\varepsilon}{2\,\mathrm{length}(\gamma)}|\zeta - z|, \quad \zeta \in B(z, r(z)). \tag{2}$$

And for each $c \in \gamma^{-1}(z)$, there is some positive radius $\delta(c) > 0$ such that $\delta(c) < \mu/2$ and $B(c, \delta(c)) \subset \gamma^{-1}(B(z, r(z)))$. Here $B(c, \delta(c))$ is simply an interval.

Because the interval $[a, b]$ is compact, its cover by the intervals $B(c, \delta(c))$ has a finite subcover $\{B(c_j, \delta_j) : j = 1, \dots, n\}$ with $c_1 < c_2 < \cdots < c_n$ and $B(c_j, \delta_j) \cap B(c_{j+1}, \delta_{j+1}) \neq \emptyset$ for $j = 1, \dots, n-1$. Take partition points

$$\begin{aligned}
t_0 &= a \in B(c_1, \delta_1)\\
t_1 &\in B(c_1, \delta_1) \cap B(c_2, \delta_2)\\
t_2 &\in B(c_2, \delta_2) \cap B(c_3, \delta_3)\\
&\vdots\\
t_{n-1} &\in B(c_{n-1}, \delta_{n-1}) \cap B(c_n, \delta_n)\\
t_n &= b \in B(c_n, \delta_n).
\end{aligned}$$

This partition has mesh less than μ.

Moving forward from the interval to the curve, let $\zeta_j = \gamma(t_j)$ for $j = 0, \dots, n$, and let $z_j = \gamma(c_j)$ for $j = 1, \dots, n$. For any $j \in \{1, \dots, n\}$, write the jth term of a telescoping sum minus a Riemann sum as the difference of two terms that we know are small,

$$F(\zeta_j) - F(\zeta_{j-1}) - f(z_j)(\zeta_j - \zeta_{j-1}) = (F(\zeta_j) - F(z_j) - f(z_j)(\zeta_j - z_j)) \\ - (F(\zeta_{j-1}) - F(z_j) - f(z_j)(\zeta_{j-1} - z_j)),$$

so that the estimate (2) gives

$$|F(\zeta_j) - F(\zeta_{j-1}) - f(z_j)(\zeta_j - \zeta_{j-1})| < \frac{\varepsilon}{2\,\mathrm{length}(\gamma)}(|\zeta_j - z_j| + |z_j - \zeta_{j-1}|).$$

Now sum over j to get

$$\left| F(\gamma(b)) - F(\gamma(a)) - \sum_j f(z_j)(\zeta_j - \zeta_{j-1}) \right| < \frac{\varepsilon}{2}\,.$$

Further, because the partition was constructed to have mesh less than μ, the estimate (1) applies,

$$\left| \sum_j f(z_j)(\zeta_j - \zeta_{j-1}) - \int_\gamma f(z)\,\mathrm{d}z \right| < \frac{\varepsilon}{2}\,.$$

So altogether,

$$\left| F(\gamma(b)) - F(\gamma(a)) - \int_\gamma f(z)\,\mathrm{d}z \right| < \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, the result follows,

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

2. The $\mathcal{C}^1$ case

If γ is a $\mathcal{C}^1$-curve and F is assumed to be $\mathcal{C}^1$ then the complex fundamental theorem of integral calculus does reduce to the real case, as follows. We have

$$\gamma = x + iy : [a, b] \longrightarrow \Omega$$

and

$$F = U + iV : \Omega \longrightarrow \mathbb{C}$$

both $\mathcal{C}^1$, and F complex-differentiable. With $f = F' = U_x + iV_x = V_y - iU_y$, compute

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b (U_x(\gamma(t)) + iV_x(\gamma(t)))(x'(t) + iy'(t))\,\mathrm{d}t \\
&= \int_a^b \begin{pmatrix} U_x(\gamma(t))x'(t) - V_x(\gamma(t))y'(t) \\ +i(U_x(\gamma(t))y'(t) + V_x(\gamma(t))x'(t)) \end{pmatrix} \mathrm{d}t \\
&= \int_a^b \begin{pmatrix} U_x(\gamma(t))x'(t) + U_y(\gamma(t))y'(t) \\ +i(V_y(\gamma(t))y'(t) + V_x(\gamma(t))x'(t)) \end{pmatrix} \mathrm{d}t \\
&= \int_a^b \big((U \circ \gamma)'(t) + i(V \circ \gamma)'(t)\big)\,\mathrm{d}t \\
&= U(\gamma(b)) - U(\gamma(a)) + i(V(\gamma(b)) - V(\gamma(a))) \\
&= F(\gamma(b)) - F(\gamma(a)).
\end{aligned}$$