BEGINNING MODULAR FORMS

This writeup gives first examples of modular forms: Eisenstein series, the discriminant, and the j-function.

1. Definitions

The *modular group* is the group of 2-by-2 matrices with integer entries and determinant 1,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

The modular group is generated by the two matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(Exercise 1). Each element of the modular group is also viewed as an automorphism (invertible self-map) of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the fractional linear transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau) = \frac{a\tau+b}{c\tau+d}, \quad \tau \in \widehat{\mathbb{C}}.$$

This is understood to mean that if $c \neq 0$ then $-d/c$ maps to ∞ and ∞ maps to a/c, and if $c=0$ then ∞ maps to ∞. The identity matrix I and its negative $-I$ both give the identity transformation, and more generally each pair $\pm\gamma$ of matrices in $\mathrm{SL}_2(\mathbb{Z})$ gives a single transformation. The group of transformations defined by the modular group is generated by the maps described by the two matrix generators,

$$\tau \mapsto \tau+1 \quad \text{and} \quad \tau \mapsto -1/\tau.$$

The *upper half plane* is

$$\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\,(\tau) > 0\}.$$

Readers with some background in Riemann surface theory—which is not necessary to read this book—may recognize $\mathcal{H}$ as one of the three simply connected Riemann surfaces, the other two being the plane $\mathbb{C}$ and the sphere $\widehat{\mathbb{C}}$. The formula

$$\mathrm{Im}\,(\gamma(\tau)) = \frac{\mathrm{Im}\,(\tau)}{|c\tau+d|^2}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(Exercise 2(a)) shows that if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$ then also $\gamma(\tau) \in \mathcal{H}$, i.e., the modular group maps the upper half plane back to itself. In fact the modular group acts on the upper half plane, meaning that $I(\tau) = \tau$ where I is the identity matrix (as was already noted) and $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$. This last formula is easy to check (Exercise 2(b)).

Definition 1.1. *Let k be an integer. A meromorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is* ***weakly modular of weight*** k *if*

$$f(\gamma(\tau)) = (c\tau+d)^k f(\tau) \quad \text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \tau \in \mathcal{H}.$$

An argument shows that if this transformation law holds when γ is each of the generators $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ then it holds for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In other words, f is weakly modular of weight k if

$$f(\tau+1) = f(\tau) \quad \text{and} \quad f(-1/\tau) = \tau^k f(\tau).$$

Weak modularity of weight 0 is simply $\mathrm{SL}_2(\mathbb{Z})$-invariance, $f \circ \gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Weak modularity of weight 2 is also natural: complex analysis relies on path integrals of differentials $f(\tau)d\tau$, and $\mathrm{SL}_2(\mathbb{Z})$-invariant path integration on the upper half plane requires such differentials to be invariant when τ is replaced by any $\gamma(\tau)$. But (Exercise 2(c))

$$d\gamma(\tau) = (c\tau + d)^{-2} d\tau,$$

and so the relation $f(\gamma(\tau))d(\gamma(\tau)) = f(\tau)d\tau$ is

$$f(\gamma(\tau)) = (c\tau + d)^2 f(\tau),$$

giving Definition 1.1 with weight $k = 2$. Weight 2 will play an especially important role later in this book since it is the weight of the modular form in the Modularity Theorem. The weight 2 case also leads inexorably to higher even weights—multiplying two weakly modular functions of weight 2 gives a weakly modular function of weight 4, and so on. Letting $\gamma = -I$ in Definition 1.1 gives $f = (-1)^k f$, showing that the only weakly modular function of any odd weight k is the zero function, but nonzero odd weight examples exist in more general contexts to be developed soon. Another motivating idea for weak modularity is that while it does not make a function f fully $\mathrm{SL}_2(\mathbb{Z})$-invariant, at least $f(\tau)$ and $f(\gamma(\tau))$ always have the same zeros and poles since the factor $c\tau + d$ on $\mathcal{H}$ has neither.

Modular forms are weakly modular functions that are also holomorphic on the upper half plane and holomorphic at ∞. To define this last notion, recall that $\mathrm{SL}_2(\mathbb{Z})$ contains the translation matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : \tau \mapsto \tau + 1,$$

for which the factor $c\tau + d$ is simply 1, so that $f(\tau + 1) = f(\tau)$ for every weakly modular function $f : \mathcal{H} \longrightarrow \mathbb{C}$. That is, weakly modular functions are $\mathbb{Z}$-periodic. Let $D = \{q \in \mathbb{C} : |q| < 1\}$ be the open complex unit disk, let $D' = D \setminus \{0\}$, and recall from complex analysis that the $\mathbb{Z}$-periodic holomorphic map $\tau \mapsto e^{2\pi i\tau} = q$ takes $\mathcal{H}$ to D'. Thus, corresponding to f, the function $g : D' \longrightarrow \mathbb{C}$ where $g(q) = f(\log(q)/(2\pi i))$ is well defined even though the logarithm is only determined up to $2\pi i\mathbb{Z}$, and $f(\tau) = g(e^{2\pi i\tau})$. If f is holomorphic on the upper half plane then the composition g is holomorphic on the punctured disk since the logarithm can be defined holomorphically about each point, and so g has a Laurent expansion $g(q) = \sum_{n\in\mathbb{Z}} a_n q^n$ for $q \in D'$. The relation $|q| = e^{-2\pi \mathrm{Im}(\tau)}$ shows that $q \to 0$ as $\mathrm{Im}\,(\tau) \to \infty$. So, thinking of ∞ as lying far in the imaginary direction, define f to be *holomorphic at* ∞ if g extends holomorphically to the puncture point $q = 0$, i.e., the Laurent series sums over $n \in \mathbb{N}$. This means that f has a *Fourier expansion*

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i\tau}.$$

Since $q \to 0$ if and only if $\mathrm{Im}\,(\tau) \to \infty$, showing that a weakly modular holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is holomorphic at ∞ doesn't require computing its

Fourier expansion, only showing that $\lim_{\mathrm{Im}(\tau)\to\infty} f(\tau)$ exists or even just that $f(\tau)$ is bounded as $\mathrm{Im}\,(\tau) \to \infty$.

Definition 1.2. *Let k be an integer. A function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is a **modular form of weight** k if*

(1) *f is holomorphic on $\mathcal{H}$,*
(2) *f is weakly modular of weight k,*
(3) *f is holomorphic at ∞.*

The set of modular forms of weight k is denoted $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$.

It is easy to check that $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ forms a vector space over $\mathbb{C}$ (Exercise 3(a)). Holomorphy at ∞ will make the dimension of this space, and of more spaces of modular forms to be defined in the next section, finite. When f is holomorphic at ∞ it is tempting to define $f(\infty) = g(0) = a_0$, but the next section will show that this doesn't work in a more general context.

The product of a modular form of weight k with a modular form of weight l is a modular form of weight $k + l$ (Exercise 3(b)). Thus the sum

$$\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$$

forms a ring, a so-called graded ring because of its structure as a sum.

2. Examples

The zero function on $\mathcal{H}$ is a modular form of every weight, and every constant function on $\mathcal{H}$ is a modular form of weight 0. For nontrivial examples of modular forms, let $k > 2$ be an even integer and define the *Eisenstein series of weight k* to be a 2-dimensional analog of the Riemann zeta function $\zeta(k) = \sum_{d=1}^{\infty} 1/d^k$,

$$G_k(\tau) = {\sum_{(c,d)}}' \frac{1}{(c\tau + d)^k}\,, \quad \tau \in \mathcal{H},$$

where the primed summation sign means to sum over nonzero integer pairs $(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. The sum is absolutely convergent and converges uniformly on compact subsets of $\mathcal{H}$ (Exercise 4(c)), so G_k is holomorphic on $\mathcal{H}$ and its terms may be rearranged. For any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, compute that

$$\begin{aligned} G_k(\gamma(\tau)) &= {\sum_{(c',d')}}' \frac{1}{\left(c'\left(\frac{a\tau+b}{c\tau+d}\right) + d'\right)^k} \\ &= (c\tau + d)^k {\sum_{(c',d')}}' \frac{1}{((c'a + d'c)\tau + (c'b + d'd))^k}. \end{aligned}$$

But as (c', d') runs through $\mathbb{Z}^2 \setminus \{(0, 0)\}$, so does $(c'a + d'c, c'b + d'd) = (c', d') \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ (Exercise 4(d)), and so the right side is $(c\tau + d)^k G_k(\tau)$, showing that G_k is weakly modular of weight k. Finally, G_k is bounded as $\mathrm{Im}\,(\tau) \to \infty$ (Exercise 4(e)), so it is a modular form.

To compute the Fourier series for G_k, continue to let $\tau \in \mathcal{H}$ and begin with the identities

$$\frac{1}{\tau} + \sum_{d=1}^{\infty} \left(\frac{1}{\tau - d} + \frac{1}{\tau + d} \right) = \pi \cot \pi\tau = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m, \quad q = e^{2\pi i\tau} \tag{1}$$

(Exercise 5—the reader who is unhappy with this unmotivated invocation of unfamiliar expressions for a trigonometric function should be reassured that it is a standard rite of passage into modular forms; but also, Exercise 6 provides other proofs, perhaps more natural, of the following formula (2)). Differentiating (1) $k-1$ times with respect to τ gives for $\tau \in \mathcal{H}$ and $q = e^{2\pi i\tau}$,

$$\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}q^m, \quad k \geq 2. \tag{2}$$

For even $k > 2$,

$$\sum_{(c,d)}{}' \frac{1}{(c\tau+d)^k} = \sum_{d\neq 0}\frac{1}{d^k} + 2\sum_{c=1}^{\infty}\left(\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)^k}\right),$$

so again letting ζ denote the Riemann zeta function and using (2) gives

$$\sum_{(c,d)}{}' \frac{1}{(c\tau+d)^k} = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{c=1}^{\infty}\sum_{m=1}^{\infty} m^{k-1}q^{cm}.$$

Rearranging the last expression gives the Fourier expansion

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \quad k > 2,\ k \text{ even}$$

where the coefficient $\sigma_{k-1}(n)$ is the arithmetic function

$$\sigma_{k-1}(n) = \sum_{\substack{m\mid n\\ m>0}} m^{k-1}.$$

Exercise 7(b) shows that dividing by the leading coefficient gives a series having rational coefficients with a common denominator. This *normalized* Eisenstein series $G_k(\tau)/(2\zeta(k))$ is denoted $E_k(\tau)$. The Riemann zeta function will be discussed further in another handout.

Since the set of modular forms is a graded ring, we can make modular forms out of various sums of products of the Eisenstein series. For example, $\mathcal{M}_8(\mathrm{SL}_2(\mathbb{Z}))$ turns out to be 1-dimensional. The functions $E_4(\tau)^2$ and $E_8(\tau)$ both belong to this space, making them equal up to a scalar multiple and therefore equal since both have leading term 1. Expanding out the relation $E_4^2 = E_8$ gives a relation between the divisor-sum functions σ_3 and σ_7 (Exercise 7(c)),

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{i=1}^{n-1}\sigma_3(i)\sigma_3(n-i), \quad n \geq 1. \tag{3}$$

The modular forms that, unlike Eisenstein series, have constant term equal to 0 play an important role in the subject.

Definition 2.1. *A **cusp form** of weight k is a modular form of weight k whose Fourier expansion has leading coefficient $a_0 = 0$, i.e.,*

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i\tau}.$$

The set of cusp forms is denoted $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$.

So a modular form is a cusp form when $\lim_{\mathrm{Im}(\tau)\to\infty} f(\tau) = 0$. The limit point ∞ of $\mathcal{H}$ is called the *cusp of* $\mathrm{SL}_2(\mathbb{Z})$ for geometric reasons that take a bit of work to explain. and a cusp form can be viewed as vanishing at the cusp. The cusp forms $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ form a vector subspace of the modular forms $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$, and the graded ring

$$\mathcal{S}(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}} \mathcal{S}_k(\mathrm{SL})_2(\mathbb{Z})$$

is an ideal in $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))$ (Exercise 3(c)).

For an example of a cusp form, let

$$g_2(\tau) = 60G_4(\tau), \qquad g_3(\tau) = 140G_6(\tau),$$

and define the *discriminant function*

$$\Delta : \mathcal{H} \longrightarrow \mathbb{C}, \qquad \Delta(\tau) = (g_2(\tau))^3 - 27(g_3(\tau))^2.$$

Then Δ is weakly modular of weight 12 and holomorphic on $\mathcal{H}$, and $a_0 = 0$, $a_1 = (2\pi)^{12}$ in the Fourier expansion of Δ (Exercise 7(d)). So indeed $\Delta \in \mathcal{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$, and Δ is not the zero function. Another writeup will show that in fact $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$ so that the only zero of Δ is at ∞.

It follows that the *modular function*

$$j : \mathcal{H} \longrightarrow \mathbb{C}, \qquad j(\tau) = 1728\frac{(g_2(\tau))^3}{\Delta(\tau)}$$

is holomorphic on $\mathcal{H}$. Since the numerator and denominator of j have the same weight, j is $\mathrm{SL}_2(\mathbb{Z})$-invariant,

$$j(\gamma(\tau)) = j(\tau), \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}),\ \tau \in \mathcal{H},$$

and in fact it is also called *the modular invariant.* The expansion

$$j(\tau) = \frac{(2\pi)^{12} + \cdots}{(2\pi)^{12}q + \cdots} = \frac{1}{q} + \cdots$$

shows that j has a simple pole at ∞ (and is normalized to have residue 1 at the pole), so it is not quite a modular form. Let ζ_3 denote the complex cube root of unity $e^{2\pi i/3}$. Easy calculations (Exercise 8) show that $g_3(i) = 0$ so that $g_2(i) \neq 0$ and $j(i) = 1728$, and $g_2(\zeta_3) = 0$ so that $g_3(\zeta_3) \neq 0$ and $j(\zeta_3) = 0$. One can further show that

$$g_2(i) = 4\varpi_4^4, \qquad \varpi_4 = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}} = 2\sqrt{\pi}\,\frac{\Gamma(5/4)}{\Gamma(3/4)}$$

and

$$g_3(\zeta_3) = (27/16)\varpi_3^6, \qquad \varpi_3 = 2\int_0^1 \frac{dt}{\sqrt{1-t^3}} = 2\sqrt{\pi}\,\frac{\Gamma(4/3)}{\Gamma(5/6)}.$$

Here the integrals are *elliptic integrals*, and Γ is Euler's *gamma function*, to be defined in a separate handout. Finally, Exercise 9 shows that the j-function surjects from $\mathcal{H}$ to $\mathbb{C}$.

Exercises.

(1) Let Γ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the two matrices $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$. Note that $\left[\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]^n \in \Gamma$ for all $n \in \mathbb{Z}$. Let $\alpha = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ be a matrix in $\mathrm{SL}_2(\mathbb{Z})$. Use the identity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b' \\ c & nc+d \end{bmatrix}$$

to show that unless $c = 0$, some matrix $\alpha\gamma$ with $\gamma \in \Gamma$ has bottom row (c, d') with $|d'| \leq |c|/2$. Use the identity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix}$$

to show that this process can be iterated until some matrix $\alpha\gamma$ with $\gamma \in \Gamma$ has bottom row $(0, *)$. Show that in fact the bottom row is $(0, \pm 1)$, and since $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]^2 = -I$ it can be taken to be $(0, 1)$. Show that therefore $\alpha\gamma \in \Gamma$ and so $\alpha \in \Gamma$. Thus Γ is all of $\mathrm{SL}_2(\mathbb{Z})$.

(2) (a) Show that $\mathrm{Im}\,(\gamma(\tau)) = \mathrm{Im}\,(\tau)\,/|c\tau + d|^2$ for all $\gamma = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$.
(b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.
(c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau + d)^2$ for $\gamma = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$.

(3) (a) Show that the set $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ of modular forms of weight k forms a vector space over $\mathbb{C}$.
(b) If f is a modular form of weight k and g is a modular form of weight l, show that fg is a modular form of weight $k + l$.
(c) Show that $\mathcal{S}_k(\mathrm{SL})_2(\mathbb{Z})$ is a vector subspace of $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ and that $\mathcal{S}(\mathrm{SL}_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))$.

(4) Let $k \geq 3$ be an integer and let $L' = \mathbb{Z}^2 \setminus \{(0, 0)\}$.
(a) Show that the series $\sum_{(c,d)\in L'}(\sup\{|c|, |d|\})^{-k}$ converges by considering the partial sums over expanding squares.
(b) Fix positive numbers A and B and let

$$\Omega = \{\tau \in \mathcal{H} : |\mathrm{Re}\,(\tau)\,| \leq A, \mathrm{Im}\,(\tau) \geq B\}.$$

Prove that there is a constant $C > 0$ such that $|\tau + \delta| > C \sup\{1, |\delta|\}$ for all $\tau \in \Omega$ and $\delta \in \mathbb{R}$.
(c) Use parts (a) and (b) to prove that the series defining $G_k(\tau)$ converges absolutely and uniformly for $\tau \in \Omega$. Conclude that G_k is holomorphic on $\mathcal{H}$.
(d) Show that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, right multiplication by γ defines a bijection from L' to L'.
(e) Use the calculation from (c) to show that G_k is bounded on Ω. From the text and part (d), G_k is weakly modular so in particular $G_k(\tau + 1) = G_k(\tau)$. Show that therefore $G_k(\tau)$ is bounded as $\mathrm{Im}\,(\tau) \to \infty$.

(5) Establish the two formulas for $\pi \cot \pi\tau$ in (1).

(6) This exercise obtains formula (2) without using the cotangent. Let $f(\tau) = \sum_{d\in\mathbb{Z}} 1/(\tau + d)^k$ for $k \geq 2$ and $\tau \in \mathcal{H}$. Since f is holomorphic (by the method of Exercise 4) and $\mathbb{Z}$-periodic and since $\lim_{\mathrm{Im}(\tau)\to\infty} f(\tau) = 0$, there is a Fourier expansion $f(\tau) = \sum_{m=1}^{\infty} a_m q^m = g(q)$ as in the section, where $q = e^{2\pi i\tau}$ and

$$a_m = \frac{1}{2\pi i}\int_\gamma \frac{g(q)}{q^{m+1}}dq$$

is a path integral once counterclockwise over a circle about 0 in the punctured disk D'.

(a) Show that

$$a_m = \int_{\tau=0+iy}^{1+iy} f(\tau)e^{-2\pi i m\tau}d\tau = \int_{\tau=-\infty+iy}^{+\infty+iy} \tau^{-k}e^{-2\pi i m\tau}d\tau \quad \text{for any } y > 0.$$

(b) Let $g_m(\tau) = \tau^{-k}e^{-2\pi i m\tau}$, a meromorphic function on $\mathbb{C}$ with its only singularity at the origin. Show that

$$-2\pi i \mathrm{Res}_{\tau=0} g_m(\tau) = \frac{(-2\pi i)^k}{(k-1)!}m^{k-1}.$$

(c) Establish (2) by integrating $g_m(\tau)$ clockwise about a large rectangular path and applying the Residue Theorem. Argue that the integral along the top side goes to a_m and the integrals along the other three sides go to 0.

(d) Let $h : \mathbb{R} \longrightarrow \mathbb{C}$ be a function such that the integral $\int_{-\infty}^{\infty} |h(x)|dx$ is finite and the sum $\sum_{d\in\mathbb{Z}} h(x+d)$ converges absolutely and uniformly on compact subsets and is infinitely differentiable. Then the *Poisson summation formula* says that

$$\sum_{d\in\mathbb{Z}} h(x+d) = \sum_{m\in\mathbb{Z}} \hat{h}(m)e^{2\pi i m x}$$

where $\hat{h}$ is the *Fourier transform* of h,

$$\hat{h}(x) = \int_{t=-\infty}^{\infty} h(t)e^{-2\pi i x t}dt.$$

We will not prove this, but the idea is that the left side sum symmetrizes h to a function of period 1 and the right side sum is the Fourier series of the left side since the mth Fourier coefficient is $\int_{t=0}^{1}\sum_{d\in\mathbb{Z}} h(t+d)e^{-2\pi i m t}dt = \hat{h}(m)$. Letting $h(x) = 1/\tau^k$ where $\tau = x + iy$ with $y > 0$, show that h meets the conditions for Poisson summation. Show that $\hat{h}(m) = e^{-2\pi m y}a_m$ with a_m from above for $m > 0$, and that $\hat{h}(m) = 0$ for $m \leq 0$. Establish formula (2) again, this time as a special case of Poisson summation. We will see more Poisson summation and Fourier analysis in connection with Eisenstein series in another handout.

(7) The *Bernoulli numbers* B_k are defined by the formal power series expansion

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Thus they are calculable in succession by matching coefficients in the power series identity

$$t = (e^t - 1)\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\binom{n}{k}B_k\right)\frac{t^n}{n!}$$

(i.e., the nth parenthesized sum is 1 if $n = 1$ and 0 otherwise) and they are rational. Since the expression

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2}\cdot\frac{e^t + 1}{e^t - 1}$$

is even, it follows that $B_1 = -1/2$ and $B_k = 0$ for all other odd k. The Bernoulli numbers will be motivated, discussed, and generalized in another handout.

(a) Show that $B_2 = 1/6$, $B_4 = -1/30$, and $B_6 = 1/42$.

(b) Use the expressions for $\pi \cot \pi\tau$ from the section to show

$$1 - 2\sum_{k=1}^{\infty} \zeta(2k)\tau^{2k} = \pi\tau \cot \pi\tau = \pi i\tau + \sum_{k=0}^{\infty} B_k \frac{(2\pi i\tau)^k}{k!}.$$

Use these to show that for $k \geq 2$ even, the Riemann zeta function satisfies

$$2\zeta(k) = -\frac{(2\pi i)^k}{k!} B_k,$$

so in particular $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$. Also, this shows that the normalized Eisenstein series of weight k

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

has rational coefficients with a common denominator.

(c) Equate coefficients in the relation $E_8(\tau) = E_4(\tau)^2$ to establish formula (3).

(d) Show that $a_0 = 0$ and $a_1 = (2\pi)^{12}$ in the Fourier expansion of the discriminant function Δ from the text.

(8) Recall that ζ_3 denotes the complex cube root of unity $e^{2\pi i/3}$. Show that $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right](\zeta_3) = \zeta_3 + 1$ so that by periodicity $g_2(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right](\zeta_3)) = g_2(\zeta_3)$. Show that by modularity also $g_2(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right](\zeta_3)) = \zeta_3^4 g_2(\zeta_3)$ and therefore $g_2(\zeta_3) = 0$. Conclude that $g_3(\zeta_3) \neq 0$ and $j(\zeta_3) = 0$. Argue similarly to show that $g_3(i) = 0$, $g_2(i) \neq 0$, and $j(i) = 1728$.

(9) This exercise shows that the modular invariant $j : \mathcal{H} \longrightarrow \mathbb{C}$ is a surjection. Suppose that $c \in \mathbb{C}$ and $j(\tau) \neq c$ for all $\tau \in \mathcal{H}$. Consider the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{j'(\tau) d\tau}{j(\tau) - c}$$

where γ is the contour containing an arc of the unit circle from $(-1+i\sqrt{3})/2$ to $(1+i\sqrt{3})/2$, two vertical segments up to any height greater than 1, and a horizontal segment. By the Argument Principle the integral is 0. Use the fact that j is invariant under $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ to show that the integrals over the two vertical segments cancel. Use the fact that j is invariant under $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ to show that the integrals over the two halves of the circular arc cancel. For the integral over the remaining piece of γ make the change of coordinates $q = e^{2\pi i\tau}$, remembering that $j'(\tau)$ denotes derivative with respect to τ and that $j(\tau) = 1/q + \cdots$, and compute that it equals 1. This contradiction shows that $j(\tau) = c$ for some $\tau \in \mathcal{H}$ and j surjects.