THE ADDITION AND DUPLICATION LAWS FOR THE WEIERSTRASS $\wp$-FUNCTION

If $\Lambda \subset \mathbb{C}$ is a lattice with associated lattice-constants g_2 and g_3, then the complex elliptic curve

$$E : y^2 = 4x^3 - g_2 x - g_3$$

inherits a group law from the complex torus $\mathbb{C}/\Lambda$ via the parametrization

$$\mathbb{C}/\Lambda \longrightarrow E, \qquad z + \Lambda \longmapsto (\wp(z), \wp'(z)).$$

Because $\wp'$ has a higher-order pole than $\wp$ at the lattice points, this notation tacitly connotes that $0 + \Lambda$ is taken to a point infinitely far away in the y-direction.

Specifically, the group law is:

- The identity element of the curve is the infinite point.
- Collinear triples on the curve sum to zero.

By analogy, the circle as a quotient parametrizes the circle as the algebraic curve $C : x^2 + y^2 = 1$ via transcendental functions,

$$\mathbb{R}/\mathbb{Z} \longrightarrow C, \qquad r + \mathbb{Z} \longmapsto (\cos(2\pi r), \sin(2\pi r)),$$

and there is a group law on the circle in consequence of the group law on the quotient. Because the parametrizing functions are transcendental, they are not easy to evaluate in general, but the geometry of the circle readily shows that the first parametrizing function satisfies an addition law and a duplication law,

$$\cos(r_1 + r_2) = \cos(r_1)\cos(r_2) - \sin(r_1)\sin(r_2),$$
$$\cos(2r) = \cos^2(r) - \sin^2(r).$$

And similarly for the second parametrizing function. The duplication law can be expressed in terms of the first parametrizing function only,

$$\cos(2r) = 2\cos^2(r) - 1.$$

Returning to the elliptic curve, we can study its group law with no direct reference to the parametrization. The results will lead to an addition law and a duplication law for the first parametrizing function $\wp$. Consider the equation of the elliptic curve and the equation of a line:

$$E : y^2 = 4x^3 - g_2 x - g_3, \qquad L : y = mx + b.$$

If a point $(x, y) \in \mathbb{C}^2$ lies on E and on L then its x-coordinate satisfies the cubic polynomial obtained by substituting $mx + b$ for y in the equation of E,

$$4x^3 - m^2 x^2 + \cdots = 0.$$

Thus, given three points collinear points (x_1, y_1), (x_2, y_2), and (x_3, y_3) on the curve, necessarily

$$x_1 + x_2 + x_3 = m^2/4,$$

where m is the slope of their line,

$$m = \begin{cases} \dfrac{y_1 - y_2}{x_1 - x_2} & \text{if } x_1 \neq x_2, \\ \dfrac{12x_1^2 - g_2}{2y_1} & \text{if } x_1 = x_2. \end{cases}$$

A slight restatement is that

$$x_3 = m^2/4 - x_1 - x_2, \quad m \text{ as above.}$$

(And also $y_3 = m(x_3 - x_1) + y_1$.)

These results translate back to the desired addition law and duplication law for the Weierstrass $\wp$-function. Because the three points on the curve are collinear, we have for some $z_1 + \Lambda, z_2 + \Lambda \in \mathbb{C}/\Lambda$,

$$\begin{aligned} (x_1, y_1) &= (\wp(z_1), \wp'(z_1)), \\ (x_2, y_2) &= (\wp(z_2), \wp'(z_2)), \\ (x_3, y_3) &= (\wp(-z_1 - z_2), \wp'(-z_1 - z_2)). \end{aligned}$$

But $\wp$ is even and $\wp'$ is odd, so that in fact

$$(x_3, y_3) = (\wp(z_1 + z_2), -\wp'(z_1 + z_2)).$$

That is,

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 - \wp(z_1) - \wp(z_2) \quad \text{if } z_1 + \Lambda \neq \pm z_2 + \Lambda,$$

$$\wp(2z) = \frac{1}{4}\left(\frac{12\wp(z)^2 - g_2}{2\wp'(z)}\right)^2 - 2\wp(z) \qquad \text{if } z \notin (1/2)\Lambda + \Lambda.$$

Again the duplication law can be expressed in terms of the first parametrizing function only,

$$\wp(2z) = \frac{1}{16} \cdot \frac{(12\wp(z)^2 - g_2)^2}{4\wp(z)^3 - g_2\wp(z) - g_3} - 2\wp(z) \quad \text{if } z \notin (1/2)\Lambda + \Lambda.$$