

THE ARZELA–ASCOLI THEOREM

Let Ω be a region in \mathbb{C} . Let $\Omega_{\mathbb{Q}}$ denote its subset of points with rational coordinates,

$$\Omega_{\mathbb{Q}} = \{x + iy \in \Omega : x, y \in \mathbb{Q}\}.$$

This subset is useful because it is small in the sense that it is countable, but large in the sense that it is dense in Ω .

Definition 0.1. A family \mathcal{F} of complex-valued functions on Ω is pointwise bounded if

$$\text{for each } z \in \Omega, \quad \sup_{f \in \mathcal{F}} \{|f(z)|\} < \infty.$$

This does not imply that any $f \in \mathcal{F}$ is bounded on Ω , as demonstrated by the family

$$\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\} \quad \text{where } f_n(z) = z/n \text{ for } z \in \mathbb{C}.$$

Nor is it implied if every $f \in \mathcal{F}$ is bounded on Ω , as demonstrated by the family

$$\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\} \quad \text{where } f_n(z) = n \text{ for } z \in \mathbb{C}.$$

Definition 0.2. A family \mathcal{F} of complex-valued functions on Ω is equicontinuous if for every $\varepsilon > 0$ and $z \in \Omega$, there exists some $\delta > 0$ such that for all $\tilde{z} \in \Omega$,

$$|\tilde{z} - z| < \delta \implies |f(\tilde{z}) - f(z)| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

The idea here is that each $f \in \mathcal{F}$ is pointwise continuous on Ω , and at each point $z \in \Omega$, given $\varepsilon > 0$, the same $\delta > 0$ works simultaneously for all $f \in \mathcal{F}$ in the definition of continuity at z .

Theorem 0.3 (Arzela–Ascoli). *Let Ω be a region in \mathbb{C} , and let \mathcal{F} be a pointwise bounded, equicontinuous family of complex-valued functions on Ω . Then every sequence $\{f_n\}$ in \mathcal{F} has a subsequence that converges to a continuous function on Ω , the convergence being uniform on compact subsets.*

Proof. Let $\{f_n\}$ be a sequence in \mathcal{F} .

First we use the given pointwise boundedness to prove that $\{f_n\}$ has a subsequence that converges on $\Omega_{\mathbb{Q}}$. The idea is a variant of Cantor’s diagonal argument. Since $\Omega_{\mathbb{Q}}$ is countable, write

$$\Omega_{\mathbb{Q}} = \{z_1, z_2, z_3, \dots\}.$$

The complex sequence

$$\{f_1(z_1), f_2(z_1), f_3(z_1), \dots\}$$

is bounded, and so it contains a convergent subsequence. Relabel the convergent subsequence as follows:

$$\{f_{1,1}(z_1), f_{1,2}(z_1), f_{1,3}(z_1), \dots\} \text{ converges.}$$

Next, the complex sequence

$$\{f_{1,1}(z_2), f_{1,2}(z_2), f_{1,3}(z_2), \dots\}$$

is again bounded, so it too contains a convergent subsequence. Relabel it:

$$\{f_{2,1}(z_2), f_{2,2}(z_2), f_{2,3}(z_2), \dots\} \text{ converges.}$$

And the complex sequence

$$\{f_{2,1}(z_3), f_{2,2}(z_3), f_{2,3}(z_3), \dots\}$$

is bounded, so it contains a convergent subsequence:

$$\{f_{3,1}(z_3), f_{3,2}(z_3), f_{3,3}(z_3), \dots\} \text{ converges.}$$

Continuing this process gives rise to an array,

$$\begin{array}{cccc} f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

The first row is a sequence of functions that converges at z_1 . The second row is a subsequence of the first row, and it converges at z_1 and at z_2 . The third row is a subsequence of the second row, and it converges at z_1 , at z_2 , and at z_3 . And so on. Consider the sequence down the diagonal,

$$\{f_{1,1}, f_{2,2}, f_{3,3}, \dots\}.$$

This is a subsequence of the original sequence $\{f_n\}$, and it converges at each $z \in \Omega_{\mathbb{Q}}$. After relabeling, we may assume that the original sequence $\{f_n\}$ converges on $\Omega_{\mathbb{Q}}$.

Next we use the given equicontinuity to prove that in fact $\{f_n\}$ converges on all of Ω . This is a typical three-epsilon argument. Let $\varepsilon > 0$ be given, and let $z \in \Omega$ be given. The equicontinuity of \mathcal{F} supplies a corresponding $\delta = \delta(\varepsilon, z) > 0$ such that for $\tilde{z} \in \Omega$,

$$|\tilde{z} - z| < \delta \implies |f(\tilde{z}) - f(z)| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

Because $\Omega_{\mathbb{Q}}$ is dense in Ω , there exists a point $z_{\mathbb{Q}} \in \Omega_{\mathbb{Q}}$ such that

$$|z_{\mathbb{Q}} - z| < \delta.$$

Because the complex sequence $\{f_n(z_{\mathbb{Q}})\}$ converges, it is Cauchy, meaning that there exists a starting index $N = N(\varepsilon, z_{\mathbb{Q}})$ such that for all integers n and m ,

$$n, m > N \implies |f_n(z_{\mathbb{Q}}) - f_m(z_{\mathbb{Q}})| < \varepsilon.$$

Consequently, the complex sequence $\{f_n(z)\}$ is Cauchy as well,

$$n, m > N \implies |f_n(z) - f_m(z)| \leq \begin{pmatrix} |f_n(z) - f_n(z_{\mathbb{Q}})| \\ + |f_n(z_{\mathbb{Q}}) - f_m(z_{\mathbb{Q}})| \\ + |f_m(z_{\mathbb{Q}}) - f_m(z)| \end{pmatrix} < 3\varepsilon.$$

Since the complex sequence $\{f_n(z)\}$ is Cauchy, it converges.

Third we prove that the pointwise limit function

$$g = \lim_n f_n : \Omega \longrightarrow \mathbb{C}$$

is continuous. Let $\varepsilon > 0$ be given, and let $z \in \Omega$ be given. The equicontinuity of \mathcal{F} supplies a corresponding $\delta = \delta(\varepsilon, z) > 0$ such that for $\tilde{z} \in \Omega$,

$$|\tilde{z} - z| < \delta \implies |f(\tilde{z}) - f(z)| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

Consider any $\tilde{z} \in \Omega$ such that $|\tilde{z} - z| < \delta$. By the pointwise convergence of $\{f_n\}$ to g , for some starting index $N = N(\varepsilon, z, \tilde{z})$ we have

$$\begin{cases} |f_n(z) - g(z)| < \varepsilon \\ |f_n(\tilde{z}) - g(\tilde{z})| < \varepsilon \end{cases} \quad \text{for all } n > N.$$

Thus

$$|g(\tilde{z}) - g(z)| \leq \begin{pmatrix} |g(\tilde{z}) - f_n(\tilde{z})| \\ + |f_n(\tilde{z}) - f_n(z)| \\ + |f_n(z) - g(z)| \end{pmatrix} < 3\varepsilon \quad \text{for all } n > N.$$

But $g(\tilde{z}) - g(z)$ is independent of n , so the “for all $n > N$ ” in the display is irrelevant: for all $\tilde{z} \in \Omega$ such that $|\tilde{z} - z| < \delta$,

$$|g(\tilde{z}) - g(z)| < 3\varepsilon.$$

That is, g is continuous at z , as desired.

The equicontinuity of \mathcal{F} and the continuity of g combine to show that also $\mathcal{F} \cup \{g\}$ is equicontinuous.

Finally we prove that the convergence of $\{f_n\}$ to g is uniform on compact subsets of Ω . Let K be such a compact set, and let $\varepsilon > 0$ be given. We seek a starting index $N = N(\varepsilon, K)$ such that for all integers n ,

$$n > N \implies |f_n(z) - g(z)| < 3\varepsilon \quad \text{for all } z \in K.$$

Let $z \in K$ be given. The equicontinuity of $\mathcal{F} \cup \{g\}$ supplies a corresponding $\delta = \delta_z = \delta(\varepsilon, z) > 0$ such that for $\tilde{z} \in K$,

$$|\tilde{z} - z| < \delta_z \implies \begin{cases} |f_n(\tilde{z}) - f_n(z)| < \varepsilon \text{ for all } n \in \mathbb{Z}^+ \\ |g(z) - g(\tilde{z})| < \varepsilon \end{cases}.$$

And because $\{f_n(z)\}$ converges to $g(z)$, there exists some $N = N_z = N(\varepsilon, z) \in \mathbb{Z}^+$ such that for all integers n ,

$$n > N_z \implies |f_n(z) - g(z)| < \varepsilon.$$

So, for all $\tilde{z} \in K$ and all integers n ,

$$\begin{cases} |\tilde{z} - z| < \delta_z \\ n > N_z \end{cases} \implies |f_n(\tilde{z}) - g(\tilde{z})| \leq \begin{pmatrix} |f_n(\tilde{z}) - f_n(z)| \\ + |f_n(z) - g(z)| \\ + |g(z) - g(\tilde{z})| \end{pmatrix} < 3\varepsilon.$$

This shows that the sequence $\{f_n\}$ converges uniformly on $B(z, \delta_z) \cap K$. So consider an open cover of the compact set K ,

$$K = \bigcup_{z \in K} B(z, \delta_z) \cap K.$$

By compactness, there exists a finite subcover,

$$K = \bigcup_{j=1}^k B(z_j, \delta_{z_j}) \cap K.$$

Define

$$N = \max\{N_{z_1}, \dots, N_{z_k}\}.$$

Then for all integers n and m , the desired condition holds,

$$n > N \implies |f_n(z) - g(z)| < 3\varepsilon \text{ for all } z \in K.$$

This completes the proof. □