PCMI 2008 Undergraduate Summer School

Lecture 9: Grassmannians I

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Question

How many lines meet 4 general lines in 3-space?

Definition

$$G(r,n) = \{r\text{-dimensional subspaces of } \mathbb{A}^n\}$$

$$\mathbb{G}_r\mathbb{P}^n = \{r\text{-planes in } \mathbb{P}^n\} = G(r+1, n+1).$$

Example

- $G(1, n+1) = \mathbb{G}_0\mathbb{P}^n = \mathbb{P}^n$
- $G(2,3) = \mathbb{G}_1\mathbb{P}^2 = (\mathbb{P}^2)^* \approx \mathbb{P}^2$
- $\mathbb{G}_{n-1}\mathbb{P}^n = (\mathbb{P}^n)^* \approx \mathbb{P}^n$
- $\mathbb{G}_1\mathbb{P}^3 = \{\text{lines in 3-space}\} = G(2,4)$

Question

Can one parametrize $G(r, n)$?

Example

$L \in G(2,4)$

$L = \mathrm{Span}\{(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)\}$

We write

$$L = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

but the representation is not unique.

$\phi \colon t \mapsto (1,0,3) + t(1,4,0) \in \mathbb{A}^3 \subset \mathbb{P}^3$

$$L = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 4 & 0 & 0 \end{pmatrix}$$

$L = \mathrm{Span}\{\vec{a_1}, \ldots, \vec{a_r}\} \in G(r, n)$

represented by the matrix A with rows $\vec{a_1}, \ldots, \vec{a_r}$.

Arbitrary representation of L:

$$MA$$

where M is an $r \times r$ invertible matrix.

$$G(r, n) = \{r \times n \text{ matrices } A \text{ of rank } r\}/(A \sim MA)$$

Open Cover

$$L = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \in G(2,4)$$

Pick two columns, say 1 and 4, and reduce:

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 4 & -3 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix}$$

Main Point: All representatives for L have the same reduction.

Note: Columns 1 and 2 would also serve, but not 3 and 4.

Let $L \in G(r,n)$. For each $1 \leq j_1 < \cdots < j_r \leq n$

$$L_{j_1,\ldots,j_r} = r \times r \text{ submatrix formed by columns } j_1, \ldots, j_r \text{ of any representative of } L$$

Definition

$$U_{j_1,\ldots,j_r} = \{L \in G(r,n) : L_{j_1,\ldots,j_r} \text{has rank } r\}$$

- $U_{j_1,\ldots,j_r}$ is well-defined.
- They cover $G(r,n)$:

$$G(r,n) = \cup_{j_1,\ldots,j_r} U_{j_1,\ldots,j_r}$$

- $U_{j_1,\ldots,j_r}$ is open:

The $r \times r$ matrix M has rank $< r$ iff $\det M = 0$.

An element of $U_{1,3,4} \subset G(3,6)$ after reduction

$$\begin{pmatrix} 1 & * & 0 & 0 & * & * \\ 0 & * & 1 & 0 & * & * \\ 0 & * & 0 & 1 & * & * \end{pmatrix}$$

$$U_{j_1, j_2, j_3} \approx \mathbb{A}^9$$

In general,

$$U_{j_1, \ldots, j_r} \approx \mathbb{A}^{r(n-r)}$$

$G(r, n)$ is an $r(n-r)$-dimensional manifold.

Example: $\mathbb{G}_1\mathbb{P}^3$ is a 4-dimensional manifold. (Common sense?)

The Plücker Embedding

Goal: embed $G(r, n)$ in projective space

Idea: Represent $L \in G(r, n)$ be its list of $r \times r$ minor determinants.

- Fix a matrix representative for $L \in G(r, n)$.
- For columns $1 \leq i_1 < \cdots < i_r \leq n$, compute $\det L_{i_1,\ldots,i_r}$.
- List these determinants for all choices of r columns:

$$\Lambda(L) = (\det L_{1,\ldots,r}, \ldots, \det L_{n-r+1,\ldots,n}).$$

Example

$$L = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

$$\Lambda(L) = (a_0 b_1 - a_1 b_0,\ a_0 b_2 - a_2 b_0,\ a_0 b_3 - a_3 b_0,\\ a_1 b_2 - a_2 b_1,\ a_1 b_3 - a_3 b_1,\ a_2 b_3 - a_3 b_2)$$

Claim: Choosing a different representative for L changes

$$\Lambda(L) = (\det L_{1,\ldots,r}, \ldots, \det L_{n-r+1,\ldots,n})$$

by a scalar multiple.

Proof of the claim.

$r \times n$ matrices A and B both represent $L \in G(r, n)$

$\implies$ $B = MA$ for some invertible $r \times r$ matrix M

$\implies$ $B_{i_1,\ldots,i_r} = M(A_{i_1,\ldots,i_r})$

$\implies$ $\det B_{i_1,\ldots,i_r} = \det M \det A_{i_1,\ldots,i_r}$

$\implies$ each term is multiplied by $\lambda = \det M$.

□

Plücker embedding

$$\begin{aligned} \Lambda \colon G(r,n) &\to \mathbb{P}^N \\ L &\mapsto (\det L_{1,\ldots,r}, \ldots, \det L_{n-r+1,\ldots,n}) \end{aligned}$$

where $N = \binom{n}{r} - 1$.

Λ is not usually surjective.

There are algebraic relations among the determinants.

CoCoA

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Use R::=Q[a[1..4],b[1..4],x[1..6]];
M:=Mat([a,b]);
M;

Mat([
  [a[1], a[2], a[3], a[4]],
  [b[1], b[2], b[3], b[4]]
])
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J:=Ideal(x-Minors(2,M));
J;

Ideal(a[2]b[1] - a[1]b[2] + x[1], a[3]b[1] - a[1]b[3] + x[2],
a[4]b[1] - a[1]b[4] + x[3], a[3]b[2] - a[2]b[3] + x[4],
a[4]b[2] - a[2]b[4] + x[5], a[4]b[3] - a[3]b[4] + x[6])
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Elim([a[1],a[2],a[3],a[4],b[1],b[2],b[3],b[4]],J);
Ideal(2x[3]x[4] - 2x[2]x[5] + 2x[1]x[6])
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$$\Lambda\colon G(r,n) \to \mathbb{P}^N$$

Coordinates on $\mathbb{P}^n$

$$\{x(i_1,\ldots,i_r) : 1 \le i_1 < \cdots < i_r \le n\}$$

conventions

- For a permutation σ,

$$x(\sigma(i_1),\ldots,\sigma(i_r)) := \mathrm{sign}(\sigma)\, x(i_1,\ldots,i_r)$$

- For any list $i_1,\ldots,i_r \in \{1,\ldots,n\}$,

$$x(i_1,\ldots,i_r) = 0 \qquad \text{if } i_k = i_\ell \text{ for some } k \text{ and } \ell$$

Plücker Relations

For each

$$\mathcal{I} : 1 \le i_1 < \cdots < i_{r-1} \le n$$
$$\mathcal{J} : 1 \le j_1 < \cdots < j_{r+1} \le n$$

define

$$P_{\mathcal{I},\mathcal{J}} = \sum_{\lambda=1}^{r+1} (-1)^{\lambda}\, x(i_1, \ldots, i_{r-1}, j_\lambda)\, x(j_1, \ldots, \widehat{j_\lambda}, \ldots, j_{r+1})$$

where $\widehat{j_\lambda}$ means omit j_λ.

Theorem

$\Lambda\colon G(r,n) \to \mathbb{P}^N$ *is one-to-one with image defined by the collection of all* $P_{\mathcal{I},\mathcal{J}}$.

Proof that $\operatorname{im}\Lambda \subseteq Z(P_{\mathcal{I},\mathcal{J}})$

$$\begin{aligned} \Lambda \colon G(r,n) &\to \mathbb{P}^N \\ L &\mapsto (\det L_{1,\dots,r}, \dots, \det L_{n-r+1,\dots,n}) \end{aligned}$$

$$P_{\mathcal{I},\mathcal{J}} = \sum_{\lambda=1}^{r+1} (-1)^\lambda \, x(i_1, \dots, i_{r-1}, j_\lambda) \, x(j_1, \dots, \widehat{j_\lambda}, \dots, j_{r+1})$$

$$P_{\mathcal{I},\mathcal{J}}(L) = \sum_{\lambda=1}^{r+1} (-1)^\lambda \det L_{i_1,\dots,i_{r-1},j_\lambda} \det L_{j_1,\dots,\widehat{j_\lambda},\dots,j_{r+1}}$$

$$\sum_{\lambda=1}^{r+1}(-1)^{\lambda} \det L_{i_1,\dots,i_{r-1},j_\lambda} \det L_{j_1,\dots,\widehat{j_\lambda},\dots,j_{r+1}}$$

$$= \sum_{\lambda=1}^{r+1}(-1)^{\lambda} \begin{vmatrix} \cdots & a_{1j_\lambda} \\ \ddots & \vdots \\ \cdots & a_{rj_\lambda} \end{vmatrix} \begin{vmatrix} \cdots & \widehat{a_{1j_\lambda}} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \widehat{a_{rj_\lambda}} & \cdots \end{vmatrix}$$

$$= \pm \sum_{\lambda=1}^{r+1}(-1)^{\lambda} \sum_{k=1}^{r}(-1)^{k} a_{kj_\lambda} \begin{vmatrix} \vdots & \ddots & \vdots \\ \widehat{a_{ki_1}} & \widehat{\cdots} & \widehat{a_{ki_{r-1}}} \\ \vdots & \ddots & \vdots \end{vmatrix} \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix}$$

$$= \pm \sum_{k=1}^{r}(-1)^{k} \begin{vmatrix} \cdots \\ \ddots \\ \cdots \end{vmatrix} \left(\sum_{\lambda=1}^{r+1}(-1)^{\lambda} a_{kj_\lambda} \begin{vmatrix} \cdots & \widehat{a_{1j_\lambda}} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \widehat{a_{rj_\lambda}} & \cdots \end{vmatrix} \right)$$

$$\pm \sum_{k=1}^{r} (-1)^k \begin{vmatrix} \cdots \\ \ddots \\ \cdots \end{vmatrix} \left(\sum_{\lambda=1}^{r+1} (-1)^\lambda a_{kj_\lambda} \begin{vmatrix} \cdots & \widehat{a_{1j_\lambda}} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \widehat{a_{rj_\lambda}} & \cdots \end{vmatrix} \right)$$

$$= \pm \sum_{k=1}^{r} (-1)^k \begin{vmatrix} \cdots \\ \ddots \\ \cdots \end{vmatrix} \begin{vmatrix} a_{kj_1} & \cdots & a_{kj_\lambda} & \cdots & a_{kj_{r+1}} \\ a_{1j_1} & \cdots & a_{1j_\lambda} & \cdots & a_{1j_{r+1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{rj_1} & \cdots & a_{rj_\lambda} & \cdots & a_{rj_{r+1}} \end{vmatrix}$$

$$= 0.$$