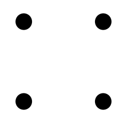
PCMI USS 2008 — *Problem Set 8*

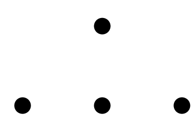
For this problem set you are asked to compute the free resolutions and Hilbert functions for all possible collections of four points in the projective plane using a computer algebra system. To get started, please read the CoCoA handout at our website.

Say the points are p_1, p_2, p_3, p_4. Consider first the case where no three lie on a line. One may show that up to a linear change of coordinates, we may assume $p_1 = (0,0,1)$, $p_2 = (1,0,1)$, $p_3 = (0,1,1)$, and $p_4 = (1,1,1)$:



(To see this, start by thinking of the points as one-dimensional subspaces of $\mathbb{A}^3$. You can of course send any three linearly independent points to any other three linearly independent points with an invertible linear map. The fact that we are interested in one-dimensional subspaces, not just points, means we are allowed to scale, and this allows one to also say something about the fourth point.) Calculate the Hilbert function and the free resolution for these points using the CoCoA commands `Hilbert`, `Res`, and `IdealOfProjectivePoints`, as indicated on the CoCoA handout.

Next, calculate the Hilbert function and free resolution in the case where exactly three of the points lie on a line.

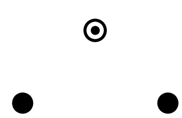


Repeat with all other special configurations of four points. (How many are there?)

Next, consider the case of four points counting "multiplicities". For instance, letting $p_1 = p_4$, we have the points $X = \{2p_1, p_2, p_3\}$. To compute the Hilbert function and resolution in this case, let I_{p_i} be the ideal for p_i for $i = 1, 2, 3$. The ideal for X is then

$$I(X) := I_{p_1}^2 \cap I_{p_2} \cap I_{p_3},$$

squaring the first ideal. (Use the `Intersection` command in CoCoA.) Note that there are two cases to consider in this case: the three points could lie on a line or not:



Find the Hilbert functions and resolutions for all collections of four points in this new sense, counting multiplicities. The most special case is $X = \{4p_1\}$: