PCMI USS 2008 *Problem Set 7*

⋆ 1. Suppose $I(X) = (f)$ with $f \in k[x_0, \ldots, x_n]$ a homogeneous polynomial of degree e. We have seen that for $d \geq e$

$$H_X(d) = \dim_k S(X)_d = \binom{n+d}{n} - \binom{n+d-e}{n}.$$

Expanding this expression as a polynomial in d, show that for $d \geq e$,

$$H_X(d) = \frac{e}{(n-1)!}\, d^{n-1} + \text{lower order terms in } d.$$

⋆ 2. Let X be the union of the two disjoint lines $\{x = y = 0\}$ and $\{z = w = 0\}$ in $\mathbb{P}^3$. Thus, $I(X) = (x, y) \cap (z, w) = (xz, xw, yz, yw)$. Calculate the Hilbert function and Hilbert polynomial for X by hand. (For $d = 0, 1, 2, 3, \ldots$, write out the monomials of degree d in four indeterminates, and set $xz = xw = yz = yw = 0$. The surviving monomials will be a basis for $S(X)_d = S_d/I(X)_d$.)

⋆ 3. Let $X \subseteq \mathbb{P}^n$ be a projective variety with Hilbert polynomial P_X. Recall that the degree of P_X is the dimension of X and that $(\dim X)!$ times the coefficient of the leading term of P_X is the degree of X. The *arithmetic genus* of X is

$$p_a(X) = (-1)^{\dim X}(P_X(0) - 1).$$

(a) Find the degree and arithmetic genus of the twisted cubic (see Lecture 7).

(b) Let $X = Z(f) \subset \mathbb{P}^2$ be a plane curve of degree d, i.e., $\deg f = d$. Show that the arithmetic genus of X is

$$p_a(X) = \frac{(d-1)(d-2)}{2}.$$

⋆ 4. Let X be the set consisting of three distinct points in $\mathbb{P}^2$. What are the possible Hilbert functions and Hilbert polynomials for X? (Recall that requiring a polynomial to vanish at a point is just a linear condition on the coefficients of the polynomial. How many linear equations will vanish at all three points? How many quadratic, etc.?)

5. Let $I \subseteq S = k[x_0, \ldots, x_n]$ be a homogeneous ideal, and suppose x_0 is not a zero divisor in S/I. Let $J = I + (x_0)$. Show that

$$H_{S/J}(d) = \Delta H_{S/I}(d) := H_{S/I}(d) - H_{S/I}(d-1).$$

(Hint: find an appropriate short exact sequence.)

6. Numerical polynomials. (From Stanley's *Enumerative Combinatorics.*)

A *numerical polynomial* is a polynomial $f \in \mathbb{Q}[t]$ such that $f(n)$ is an integer for each integer n. Of course, every $f \in \mathbb{Z}[t]$ is a numerical polynomial. The polynomial $\frac{1}{2}n^2 - \frac{1}{2}n$ is an example of a numerical polynomial with coefficients that are not integers. Hilbert functions are examples of numerical polynomials.

For any function $f\colon \mathbb{Z} \to \mathbb{C}$, not just polynomial functions, define the *first difference operator*, Δ, by

$$\Delta f(n) = f(n+1) - f(n).$$

Thus, $\Delta f\colon \mathbb{Z} \to \mathbb{C}$, too. The *k-th difference operator* is $\Delta^k = \Delta(\Delta^{k-1} f)$.

(a) Show

$$\Delta^k f(n) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(n+i)$$

and hence,

$$\Delta^k f(0) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(i)$$

(Hint: Define the *shift operator* E by $Ef(n) = f(n+1)$. Then $\Delta = E - 1$ where 1 denotes the identity operator, $1f = f$. Substitute for Δ^k and expand.)

(b) Show $f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0)$. (Hint: $f(n) = E^n f(0)$ and $E = 1 + \Delta$.)

(c) Show f is a polynomial of degree at most d over $\mathbb{C}$ iff $\Delta^{d+1} f = 0$ (equivalently, $\Delta^d f$ is constant or $\Delta^k f = 0$ for all $k > d$). (Hint: induction.)

(d) If f is a polynomial of degree d, then

$$f(t) = \sum_{k=0}^{d} \Delta^k f(0) \binom{t}{k}$$

where, by definition,

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}.$$

Note that $f\colon \mathbb{Z} \to \mathbb{Z}$ iff $\Delta^k f(0) \in \mathbb{Z}$ for $0 \le k \le d$. In particular, **every numerical polynomial of degree d is an integer combination of $\{\binom{t}{k}\}_{k=0}^{d}$.**

(e) Using the formula just given, find the polynomial f, of degree 4 whose first 5 values $f(0), f(1), f(2), f(3), f(4)$, are

$$1 \quad 0 \quad 3 \quad 34 \quad 141 \quad 396.$$

(Hint: under the given row of numbers, write their first differences in a row. Then take the first differences of the row just constructed. Continue. Along the left-hand edge, you will be computing $\Delta^k f(0)$.)