⋆ 1. Let $I = (x + z^2, y + z^2)$. Show that the homogenization of I in $k[w, x, y, z]$ is not obtained by simply homogenizing the given generators of I.

⋆ 2. Let $f = y - x^2$ and $g = z - x^3$. The affine twisted cubic is $C = Z(f, g)$. Its projective closure is $\bar{C} = Z(wy - x^2, zw - xy, y^2 - xz)$. Characterize the points in $Z(f^h, g^h) \setminus \bar{C}$.

⋆ 3. Let $f = k[x, y, z]$ be a homogeneous polynomial and let g be homogenization of $f(x, y, 1)$ with respect to z. Give an example for which $f \neq g$.

⋆ 4. Show that if $X = Z(I) \subset \mathbb{A}^n$, then $Z(I^h) \subset \mathbb{P}^n$ is the smallest projective algebraic set containing X. (Hint: for a homogeneous polynomial f, we have that $x_0^e(f(1, x_1, \ldots, x_n))^h = f(x_0, x_1, \ldots, x_n)$ for some e.)

5. Let $f \in \mathbb{C}[x, y]$ be a polynomial of degree 2. The algebraic set $X = Z(f)$ is a *plane conic*. Show that X is a circle iff its points at infinity are exactly the *circular points at infinity*: $(i, 1, 0)$, $(1, i, 0)$.

⋆ 6. Find the point at ∞ of the plane curve $y = x^3$. Change coordinates to see this point and find that this otherwise smooth curve has a singularity at ∞.

7. Dual curves.

   Let C be a plane curve: $C = Z(f)$ where $f \in k[x, y]$, and let $\bar{C}$ be its projective closure. Thus, $\bar{C} = Z(\bar{f})$ where $\bar{f} = f^h \in k[x, y, z]$ is the homogenization of f. Recall that, letting $(\mathbb{P}^2)^*$ denote the collection of lines in the projective plane, $\mathbb{P}^2$, there is a one-to-one correspondence

$$\begin{array}{ccc} (\mathbb{P}^2)^* & \leftrightarrow & \mathbb{P}^2 \\ ax + by + cz & \leftrightarrow & (a, b, c) \end{array}$$

   We then just identify $(\mathbb{P}^2)^*$ and $\mathbb{P}^2$, referring interchangeably to $ax + by + cz$ or (a, b, c).

   (a) Compute the equation for the tangent line to C at a point $p \in C$. Since C is a level set of f, the tangent line will be the line passing through p and perpendicular to the gradient ∇f.

   (b) Prove Euler's formula: if $g \in k[x_0, \ldots, x_n]$ is homogeneous of degree d, then

$$\begin{aligned} (x_0, \ldots, x_n) \cdot \nabla g &= \sum_{i=0}^{n} x_i \frac{\partial g}{\partial x_i} \\ &= \deg(g)\, g. \end{aligned}$$

(c) Show that the homogenization of the equation of the tangent line in part (a) gives the point

$$\nabla \bar{f}(p) = \left(\frac{\partial \bar{f}}{\partial x}(p), \frac{\partial \bar{f}}{\partial y}(p), \frac{\partial \bar{f}}{\partial z}(p) \right) \in (\mathbb{P}^2)^*.$$

[Hint: Euler's formula.] As usual, identify the point $p = (a, b) \in \mathbb{A}^2$ with $(a, b, 1) \in \mathbb{P}^2$.

(d) Let $\bar{C}^\circ$ denote the nonsingular points of $\bar{C}$, i.e., those points $p \in \bar{C}$ such that $\nabla \bar{f}(p) \neq 0$. Define

$$\begin{aligned} \partial : \bar{C}^\circ &\rightarrow (\mathbb{P}^2)^* = \mathbb{P}^2 \\ p &\mapsto \nabla \bar{f}(p) \end{aligned}$$

Definition. The *dual curve* to C, denoted $\hat{C}$, is the smallest projective algebraic set containing the closure of the image of ∂, i.e., the projective closure of $\operatorname{im} \partial$.

(e) Compute the dual curve to $y^2 = x^3$ by eliminating x, y, z from the equations

$$\nabla \bar{f} = (u, v, w), \quad y^2 = x^3$$

where $\bar{f} = zy^2 - x^3$. To which point on the dual does the cusp point $(0, 0, 1)$ on f correspond? (Hint: parametrize the curve by $t \mapsto (t^2, t^3, 1) \in \mathbb{P}^2$, then compose with the duality map ∂).

(f) Suppose there is line tangent to a plane curve at two points. What can you say about the dual curve?

(g) Let $C = Z(f)$ be a general plane conic, writing

$$f(x, y, z) = a_0 x^2 + 2a_1 xy + 2a_2 xz + a_3 y^2 + 2a_4 yz + a_5 z^2.$$

i. Find a symmetric 3×3 matrix M such that

$$f = \begin{pmatrix} x & y & z \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We say that M is the matrix corresponding to the conic f.

ii. Show that the dual to C is the conic defined by M^{-1}. (Hint: Eliminate x, y, and z from the system of equations $(u, v, w) = \nabla f$ and $f = 0$).