# PCMI 2008 Undergraduate Summer School

## Lecture 5: Projective Space I

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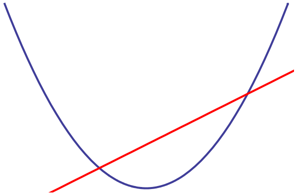
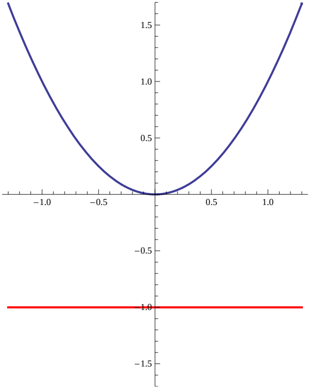
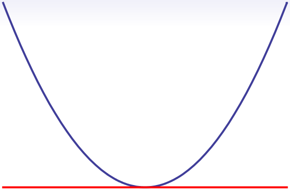
Every line meets the parabola $y = x^2$ in 2 points.





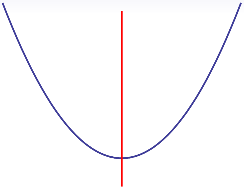

$$\left.\begin{array}{rcl} y & = & x^2 \\ y & = & -1 \end{array}\right\} \implies (x, y) = (\pm i, -1)$$



$$\left.\begin{array}{rcl} y & = & x^2 \\ y & = & 0 \end{array}\right\} \implies (x, y) = (0, 0)$$

$$k[x, y]/(y, y - x^2) = k[x, y]/(y, x^2) \approx k[x]/(x^2) = \mathrm{Span}_k\{1, x\}$$

$$k[x, y]/(x, y) \approx k$$



$$\left.\begin{array}{rcl} y & = & x^2 \\ x & = & 0 \end{array}\right\} \implies (x, y) = (0, 0)$$

$$k[x, y]/(x, y - x^2) = k[x, y]/(x, y) = k$$

Disturbing.

- Projective geometry is related to the idea of perspective from art.
- Curves are projectively equivalent if they are shadows of the same curve.
- The “points” of projective geometry are all lines through a special point.

## Definition

$$\begin{aligned} \mathbb{P}_k^n &= \{\text{lines through the origin in } \mathbb{A}_k^{n+1}\} \\ &= \{\text{one-dimensional subspaces of } k^{n+1}\} \end{aligned}$$

## Note

A point in $\mathbb{P}_k^n$ is a line in affine $(n+1)$-space.

## Question

What kinds of polynomials vanish on subsets of projective space?

## Definition

A polynomial is homogeneous if each of its monomials has the same degree.

## Example

homogeneous: $3x^2yz - y^3z + 5z^4$

non-homogeneous: $x^2 - 4xy^4 + z^9$

Every polynomial is the sum of its homogeneous components:

$$f = f_0 + f_1 + \cdots + f_d$$

with $f_i$ homogeneous of degree $i$.

## Example

$$\underbrace{5}_{0} + \underbrace{3x + 2y}_{1} + \underbrace{2xy + z^2}_{2} + \underbrace{x^3}_{3}$$

## Proposition

*Over an infinite field, $f$ vanishes on a line through the origin iff each $f_i$ does.*

Suppose $f$ is homogeneous of degree $d$.

For all $\lambda \in k$ and $p \in \mathbb{A}^{n+1}$, we have

$$f(\lambda p) = \lambda^d f(p).$$

Hence, for $\lambda \neq 0$,

$$f(p) = 0 \iff f(\lambda p) = 0.$$

The point:

$$Z(f) \subset \mathbb{P}^n \text{ makes sense.}$$

Let $I \subseteq S = k[x_0, \ldots, x_n]$ be an ideal.

## Definition

$I$ is homogeneous if it is generated by homogeneous polynomials.

## Example

$I = (yz - x^2, y^2z - x^3 - xz^2)$

## Proposition

*$I$ is homogeneous iff it contains the homogeneous components of each of its elements.*

# Projective algebraic sets

$S = k[x_0, \ldots, x_n], \quad I \subset S$ homogeneous

**projective algebraic set**

$$Z(I) = \{p \in \mathbb{P}^n_k : f(p) = 0 \text{ for all homog. } f \in I\}$$

**ideal of $X \subseteq \mathbb{P}^n_k$**

$$I(X) = (f \in S : f \text{ homog., } f(p) = 0 \text{ for all } p \in X)$$

## Projective correspondence

| Algebra | | Geometry |
|---|---|---|
| homogeneous ideals of $S$ | $\longleftrightarrow$ | subsets of $\mathbb{P}^n$ |
| $I$ | $\rightarrow$ | $Z(I)$ |
| $I(X)$ | $\leftarrow$ | $X$ |

## As before

$$I(Z(J)) \supseteq J \qquad Z(I(X)) \supseteq X$$

$$Z(I(Z(J))) = Z(J) \qquad I(Z(I(X))) = I(X)$$

## Caution!

$$Z(1) = Z(x_0, \ldots, x_n) = \emptyset \subset \mathbb{P}^n$$

## Definition

$\mathfrak{m} := (x_0, \dots, x_n)$ is the irrelevant ideal of $S$.

## Theorem (Projective Nullstellensatz)

*If $k$ is algebraically closed and $J \subset S$ is a homogeneous ideal,*

- $Z(J) = \emptyset \in \mathbb{P}^n \iff \operatorname{rad} J \supseteq \mathfrak{m}$.
- $Z(J) \neq \emptyset \in \mathbb{P}^n \implies I(Z(J)) = \operatorname{rad} J$;

Note:

$$Z(J) = \emptyset \iff \operatorname{rad}(J) = S \quad \text{or} \quad \mathfrak{m}.$$

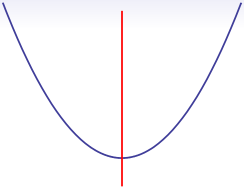
# Projective correspondence

For $k$ algebraically closed, there is a one-to-one correspondence:

| Algebra | | Geometry |
|---|---|---|
| homogeneous radical ideals $\neq \mathfrak{m}$ | $\longleftrightarrow$ | algebraic subsets of $\mathbb{P}^n$ |
| $I$ | $\rightarrow$ | $Z(I)$ |
| $I(X)$ | $\leftarrow$ | $X$ |

**projective varieties**

prime ideals $\leftrightarrow$ irreducible projective algebraic sets



$$Z(y - x^2) \subset \mathbb{A}^2 \implies ? \subset \mathbb{P}^2$$

$$Z(y - x^2) \subset \mathbb{A}^2 \implies Z(zy - x^2) \subset \mathbb{P}^2$$

$$Z(x) \subset \mathbb{A}^2 \implies Z(x) \subset \mathbb{P}^2$$

$$\left.\begin{array}{rcl} zy & = & x^2 \\ x & = & 0 \end{array}\right\} \implies (x, y, z) \in \{(0,0,1), (0,1,0)\}$$

### Theorem (Bezout's theorem)

*Let $X = Z(f)$ and $Y = Z(g)$ be distinct curves in $\mathbb{P}^2$ over an algebraically closed field. Then, the number of points in their intersection of $X$, counting multiplicities, is*

$$\sharp(X \cap Y) = (\deg f)(\deg g).$$