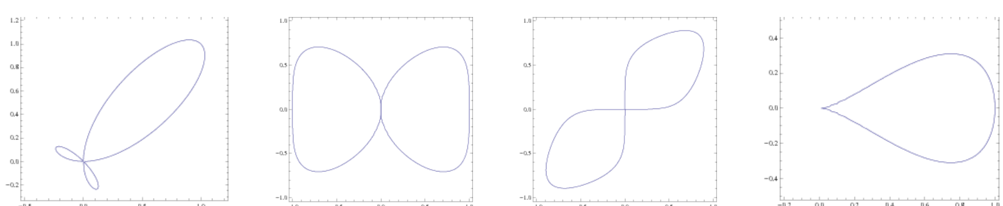
PCMI USS 2008 *Problem Set 4*

1. Find the (Krull) dimension of the following rings:
   - ⋆ (a) k, where k is a field.
   - ⋆ (b) $\mathbb{Z}$.
   - (c) $\mathbb{Z}[x]$.
   - (d) The Gaussian integers, $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$.

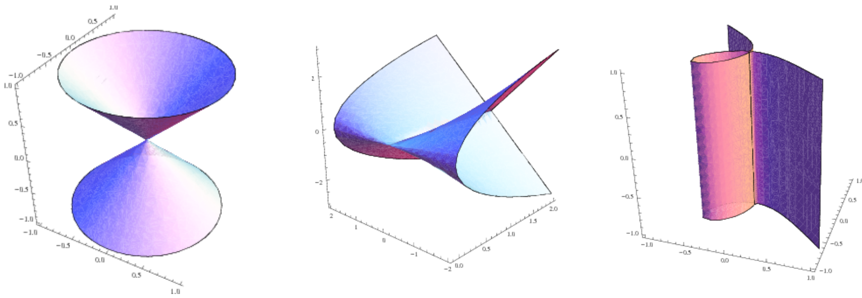
2. Find a ring which is not a field with (Krull) dimension equal to zero.

⋆ 3. Prove that the plane nodal cubic curve defined by $y^2 = x^3 + x^2$ has dimension 1 by examining its quotient field. (Hint: recall the parametrization of this curve given in an earlier problem set.)

⋆ 4. (From Hartshorne's "Algebraic geometry".) Show that the origin is a singular point for each of the following curves in $\mathbb{A}^2_k$ assuming the characteristic of k is not 2. Match up the curves with the sketches given below.
   - (a) $x^2 = x^4 + y^4$;
   - (b) $xy = x^6 + y^6$;
   - (c) $x^3 = y^2 + x^4 + y^4$;
   - (d) $x^2y + xy^2 = x^4 + y^4$.



⋆ 5. (From Hartshorne's "Algebraic geometry".) Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^3_k$, again assuming the characteristic of k is not 2. Match up the surfaces with the sketches given below.
   - (a) $xy^2 = z^2$;
   - (b) $x^2 + y^2 = z^2$;
   - (c) $xy + x^3 + y^3 = 0$.



6. Find the singularities of the quartic

$$50xy((x+y-1)^2 - xy) - (x+y)^2(x+y-1)^2 = 0$$

and sketch the curve. (Hint: Using Mathematica/Maple/Matlab/Sage would make this problem a lot easier.)

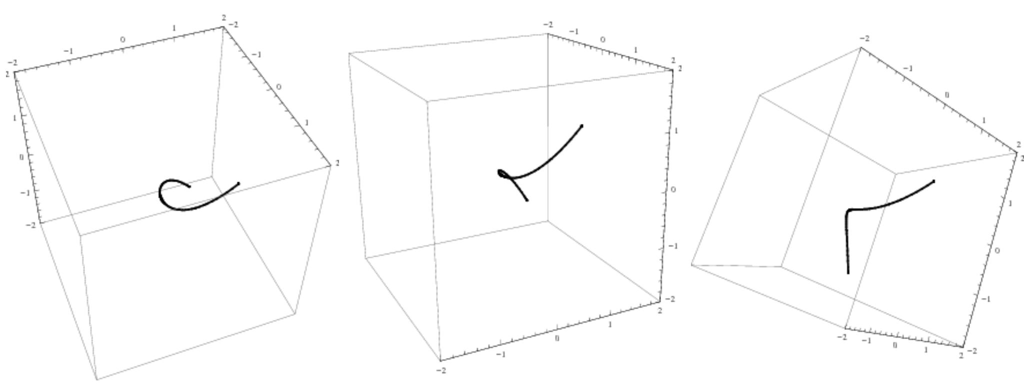
7. The *twisted cubic* is the curve, T, given parametrically by $x = t$, $y = t^2$, and $z = t^3$.

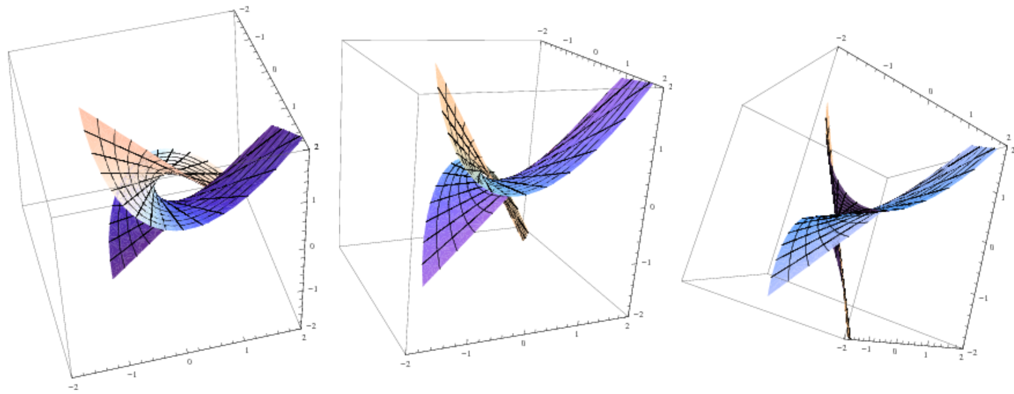
   (a) The *osculating developable*, $\operatorname{Osc}(T)$, of the twisted cubic is the union of all tangent lines to T. Parametrize $\operatorname{Osc}(T)$ with a function $h(s,t)$ such that $h(0,t) = (t, t^2, t^3)$.

   (b) The surface $\operatorname{Osc}(T)$ is defined by the equation

   $$3x^2y^2 - 4x^3z - 4y^3 + 6xyz - z^2 = 0.$$

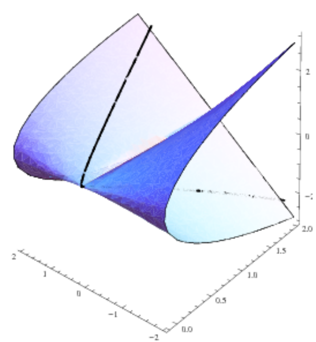
   Show that $\operatorname{Osc}(T)$ is singular exactly along the twisted cubic (which, of course, lies on the surface).





☐ 7. Consider the curve X given parametrically by $x = t^3$, $y = t^4$, $z = t^5$.

   (a) Show that $I(X) = (-x^3+yz, -y^2+xz, x^2y-z^2)$. By the way, note that the curve sits on the surface defined by $x^2y = z^2$.



   (b) Show that $I(X)$ cannot be generated by two elements.

   (c) Calculate the singularities of X.

9. Consider the curve X defined by cuspidal cubic defined by $y^2 = x^3$.

   ⋆ (a) Show that X is singular at $(0,0)$ using Zariski's definition, i.e., by calculating the vector space dimension of $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal for the point $(0,0)$ in the coordinate ring for X.

   ☐ (b) Show that X is nonsingular at $(1,1)$ using Zariski's definition. (Hints: Find the equation for the tangent line to X at the point $(1,1)$ and use it to show that $x - 1 = \alpha(y-1)$ for some constant α. You will need to show the equation is in the square of the maximal ideal for $(1,1)$ in $A(X)$.)

10. Show that $\mathbb{Z}$ is regular (nonsingular).

11. Find a curve in $\mathbb{A}^2_k$ for which every point is singular.

12. Show that the singular points on a variety form an algebraic set.

⋆ 13. Let $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$, and let $M_p = (x_1 - a_1, \ldots, x_n - a_n) \subset k[x_1, \ldots, x_n]$. During the lecture, we defined the mapping

$$\begin{array}{rcl} \nabla_p \colon M_p & \to & k^n \\ f & \mapsto & \nabla f(p) \end{array}$$

(a) Prove that ∇_p is onto and $\ker \nabla_p = M_p^2$. Hence, ∇_p induces an isomorphism $M_p/M_p^2 \approx k^n$. (Hint: for the kernel, consider Taylor series about the point p.)

(b) Let X be an algebraic set in $\mathbb{A}_k^n$ with ideal $I(X) = (f_1, \ldots, f_m)$. Suppose $p \in X$. Show that

$$\nabla_p(I(X)) = \mathrm{Span}_k\{\nabla f_1(p), \ldots, \nabla f_m(p)\}.$$

Thus, the definition of nonsingularity given during the lecture does not depend on the choice of generators of $I(X)$.