PCMI 2008 Undergraduate Summer School

Lecture 4: Dimension, singularities.

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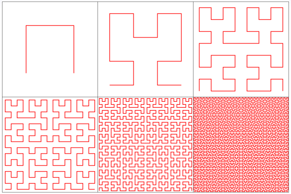
Dimension



Cantor to Dedekind (1877):

> *Your latest reply about our work was so unexpected and so novel that in a manner of speaking I will not be able to attain a certain composure until I have had from you, my very dear friend, a decision on its validity. As long as you have not confirmed it, I can only say:* I see it but I don't believe it *. . . the distinction between domains of* different *dimensions must be sought for in quite another way than by the characteristic number of independent coordinates.*

It's not the number of equations.

The curve X given parametrically by

$$x = t^3, \qquad y = t^4, \qquad z = t^5$$

has ideal

$$I(X) = (-x^3 + yz, -y^2 + xz, x^2y - z^2).$$

Challenge!

Prove this and show that $I(X)$ needs at least three generators.

Assumptions

- k is algebraically closed.
- $X \subseteq \mathbb{A}_k^n$ is a variety.
- Thus, $I(X)$ is prime and

$$A(X) = R/I(X)$$

is a domain.

Goal

Define $\dim X$ using its coordinate ring $A(X)$.

Dimension: Version 1

Quotient fields

Definition

The quotient field of a domain A is its field of fractions

$$K(A) = \left\{ \frac{f}{g} : g \neq 0 \right\}$$

Example

$$K(\mathbb{Z}) = \mathbb{Q}$$

$$K(k[x]) = k(x)$$

Algebraic independence

Let $k \subset K$ be fields, and let $S \subset K$.

Definition

The set S is algebraically independent over k if for every subset $\{s_1, \ldots, s_n\} \subseteq S$ with $n \geq 1$ and every nonzero polynomial $f(x_1, \ldots, x_n)$ with coefficients in k, we have $f(s_1, \ldots, s_n) \neq 0$.

Example

- $x, y \in k(x, y)$ are algebraically independent over k.
- $\pi \in \mathbb{R}$ is algebraically independent over $\mathbb{Q}$. (Lindemann, 1882)
- $x^2, x^4 + 3x^2 - 1 \in k(x)$ are not algebraically independent over k.

Transcendence degree

Definition

The transcendence degree of K over a subfield k is the maximal size of an algebraically independent subset of K over k.

Example

$\mathrm{tr.deg}(k(x_1, \ldots, x_n)) = n$

Definition

The dimension of the variety X is the transcendence degree of its quotient field, $K(X) := K(A(X))$.

Example

- $\dim(\mathbb{A}^n_k) = n$

$$A(\mathbb{A}^n_k) = k[x_1, \ldots, x_n], \qquad K(\mathbb{A}^n_k) = k(x_1, \ldots, x_n).$$

- $X = Z(y^2 - x^3) \subset \mathbb{A}^2_k$.

$$\begin{array}{rcl} k[x,y]/(y^2 - x^3) & \approx & k[t^2, t^3] \\ f(x,y) & \mapsto & f(t^2, t^3) \end{array}$$

$K(X) \approx k(t) \Longrightarrow \dim(X) = 1.$

Dimension: Version 2 (Krull, 1937)

Let A be any ring.

Definition

The height of a prime ideal $\mathfrak{p} \subseteq A$ is the largest d such that there exists a chain of distinct prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d = \mathfrak{p}$$

Definition

$$\dim A = \sup\{\mathrm{height}(\mathfrak{p}) : \mathfrak{p} \text{ a prime of } A\}.$$

Definition

$$\dim X = \dim A(X).$$

Example

$X = \mathbb{A}^n, \quad A(X) = k[x_1, \ldots, x_n]$

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_n)$$

Example

$X = Z(z - x^2 - y^2) \subset \mathbb{A}^3$.

In $k[x, y, z]$,

$$(z - x^2 - y^2) \subset (z - x^2 - y^2, x) \subset (z - x^2 - y^2, x, y).$$

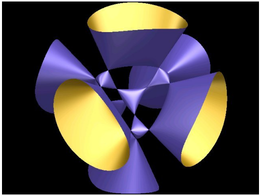
In $k[x, y, z]$,

$$(z - x^2 - y^2) \subset \underbrace{(z - x^2 - y^2, x)}_{(z-y^2,x)} \subset \underbrace{(z - x^2 - y^2, x, y)}_{(x,y,z)}.$$

In $A(X) = k[x, y, z]/(z - x^2 - y^2)$,

$$(0) \subset (x) \subset (x, y).$$

Singularities



Gradient

Definition

The gradient of a polynomial $f \in k[x_1, \ldots, x_n]$ is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Fact from multivariable calculus:

Suppose p lies on the level set defined by $f = 0$.

Then $\nabla f(p)$ is perpendicular to the level set $f = 0$ at the p.

$p \in X \subseteq \mathbb{A}^n$ a variety with $I(X) = (f_1, \ldots, f_m)$.

Definition

The variety X is nonsingular at p if

$$\dim \mathrm{Span}_k\{\nabla f_1(p), \ldots, \nabla f_m(p)\} = n - \dim X.$$

Question: What about arbitrary rings?

Is it possible to define nonsingularity without reference to polynomials and derivatives?

Exercise

$M_p = (x_1 - a_1, \ldots, x_n - a_n) \subset R = k[x_1, \ldots, x_n]$

Define

$$\begin{aligned} \nabla_p : M_p &\to k^n \\ f &\mapsto \nabla f(p) \end{aligned}$$

Suppose $p \in X$. Then

- If $I(X) = (f_1, \ldots, f_m)$,

$$\nabla_p(I(X)) = \mathrm{Span}_k\{\nabla f_1(p), \ldots, \nabla f_m(p)\}.$$

- X is nonsingular at p iff $\dim_k \nabla_p(I(X)) = n - \dim X$.
- ∇_p onto and $\ker \nabla_p = M_p^2$, hence, ∇_p induces

$$M_p/M_p^2 \approx k^n.$$

Important aside

Since X is nonsingular at p iff $\dim_k \nabla_p(I(X)) = n - \dim X$, the notion of singularity does not depend on the choice of generators $f_1, \ldots, f_m$ for I.

Let $\mathfrak{m}_p = M_p \bmod I(X)$.

- $\mathfrak{m}_p/\mathfrak{m}_p^2 \approx M_p/(I(X) + M_p^2)$.
- An exact sequence of vector spaces:

$$\begin{array}{ccccccccc}
0 \longrightarrow & (I(X)+M_p^2)/M_p^2 & \longrightarrow & M_p/M_p^2 & \longrightarrow & M_p/(I(X)+M_p^2) & \longrightarrow 0 \\
& \nabla \downarrow \wr & & \nabla \downarrow \wr & & \downarrow \wr & \\
& \nabla_p(I(X)) & & k^n & & \mathfrak{m}_p/\mathfrak{m}_p^2 &
\end{array}$$

- rank-nullity $\Longrightarrow$

$$\begin{aligned}
\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 &= n - \dim_k \nabla_p(I(X)) \\
&= \dim X \quad \text{iff } X \text{ is nonsingular at } p
\end{aligned}$$

Definition

A ring A is regular (nonsingular) if for each maximal ideal $\mathfrak{m} \subset A$, we have

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$$

where $k = A/\mathfrak{m}$.