$\star$ = more important problem $\square$ = challenge

Again, k denotes a field, and $R = k[x_1, \ldots, x_n]$.

$\star$ 1. Intersections and products of ideals. Let I and J be ideals of an arbitrary ring B.

(a) Show that $I \cap J \supseteq IJ$. Given an example to show that the inclusion can be proper.

(b) Show that if $I + J = B$, then $I \cap J = IJ$.

(c) If $B = R = k[x_1, \ldots, x_n]$, show that $Z(I \cap J) = Z(IJ)$.

Thus, from this exercise and results from Problem Set 1, you should see that

$$Z(I_1) \cup \cdots \cup Z(I_m) = Z(I_1 \cap \cdots \cap I_m) = Z(I_1 \cdots I_m)$$

and

$$\cap_\alpha Z(I_\alpha) = Z(\cup_\alpha I_\alpha) = Z(\textstyle\sum_\alpha I_\alpha),$$

where α runs over an arbitrary index set. The notation $\sum_\alpha I_\alpha$ denotes the collection of finite sums of the form $\sum_\alpha f_\alpha$ with $f_\alpha \in I_\alpha$.

$\star$ 2. The Nullstellensatz.

For the following problems, assume that k is algebraically closed.

(a) Let $f_1, \ldots, f_m \in R$. Show that the system of equations $f_1 = \cdots = f_m = 0$ has no solutions iff 1 is an R-linear combination of the f_i:

$$1 = \sum_{i=1}^{n} g_i f_i$$

for some polynomials $g_i \in R$. The implication still runs in one direction, even if k is not algebraically closed. Which one?

(b) Show that an ideal $I \subset R$ is maximal iff $I = (x_1 - a_1, \ldots, x_n - a_n)$ for some $(a_1, \ldots, a_n) \in \mathbb{A}^n_k$. Show by example that this result does not hold if k is not algebraically closed. We will go over this problem during the lecture.

(c) If I is an ideal of R, not equal to R, then $Z(I) \neq \emptyset$. (This result is called the "weak Nullstellensatz".) Again, show this result does not hold if k is not algebraically closed.

$\square$ 3. Is 1 in the ideal $(x^2 + y - 3, xy^2 + 2x, y^3)$? Does the answer depend on k?

⋆ 4. Give the decomposition of $X = Z(z - xy, z^2 - xy)$ into irreducibles over $\mathbb{R}$. List the corresponding prime ideals.

5. (a) Show that it possible for $I(\mathbb{A}_k^n) \neq (0)$.

   ⋆ (b) Show that $I(\mathbb{A}_k^n) = (0)$ if k is infinite. (Hint: induction on n.)

6. Cartesian products. Let $X \subset \mathbb{A}^n$ and $Y \subset A^m$ be algebraic sets.

   (a) Show that

   $$X \times Y := \{(p_1, \ldots, p_n, q_1, \ldots, q_m) \in \mathbb{A}^{n+m} : (p_1, \ldots, p_n) \in X \text{ and } (q_1, \ldots, q_m) \in Y\}$$

   is an algebraic set.

   □ (b) Show that if X and Y are varieties, so is $X \times Y$.

7. Let B be a ring. If $B[x]$ is Noetherian, does it follow that B is Noetherian? (Hint: Consider the mapping $\phi\colon B[x] \to B$ sending $f(x) \to f(0)$. To show an ideal I of B is finitely generated, look at $\phi^{-1}(I)$.)

8. Show that $\{(t, \cos(t)) : t \in \mathbb{R}\}$ is not an algebraic set. (What can you say about a polynomial $f(x, y)$ such that $f(t, \cos(t)) = 0$ for all $t \in \mathbb{R}$?)