$\star$ = more important problem, $\square$ = challenge

In the following, k denotes a field, and $R = k[x_1, \ldots, x_n]$.

1. Give a description of $Z(z - xy, z^2 - xy)$ over the real numbers: solve the equations and draw the picture.

$\star$ 2. Verify the following basic properties:

   (a) If $S \subseteq R$ and I is the ideal generated by S, then $Z(S) = Z(I)$.

   (b) If $S \subseteq T \subseteq R$, then $Z(S) \supseteq Z(T)$.

   (c) If $X \subseteq Y \subseteq \mathbb{A}^n$, then $I(X) \supseteq I(Y)$.

   (d) For $S \subseteq R$ and $X \subseteq \mathbb{A}^n$, we have

      i. $I(Z(S)) \supseteq S$;

      ii. $Z(I(X)) \supseteq X$;

      iii. $Z(I(Z(S))) = Z(S)$ (so if X is an algebraic set, then $Z(I(X)) = X$);

      iv. $I(Z(I(X))) = I(X)$ (so if J is the ideal of an algebraic set, then $I(Z(J)) = J$);

$\star$ 3. (a) Show that if X and Y are algebraic subsets of $\mathbb{A}^n$, then $X = Y$ iff $I(X) = I(Y)$.

   (b) Let $X \subset \mathbb{A}^n$ be an algebraic set and let $p \in \mathbb{A}^n \setminus X$. Show there exists $f \in R$ such that $f(q) = 0$ for all $q \in X$ and $f(p) = 1$. (Hint: let $Y = X \cup \{p\}$ and apply the first part of this problem.)

4. Show that the following are algebraic sets:

   (a) $\{(t, t^2, t^3) : t \in k\}$.

   (b) $\{(\cos t, \sin t) : t \in \mathbb{R}\}$.

5. What are the algebraic subsets of $\mathbb{A}^1_k$?

6. Radical ideals.

   (a) Show that the radical of an ideal is an ideal. (Hint: if f^s and g^t are elements of an ideal $I \subseteq R$, then $(f + g)^{s+t} \in I$.)

   (b) Show that $I(X)$ is a radical ideal for all $X \subseteq \mathbb{A}^n$.

   (c) An ideal $I \subset R$ is a *prime ideal* if $fg \in I$ implies f or g is an element of I. Show that a prime ideal is radical.

   (d) If $f \in R$ is irreducible, does it follow that the ideal (f) is radical?

(e) For any ideal $J \subseteq R$, we have $Z(J) = Z(\text{rad}(J))$ and $\text{rad}(J) \subseteq I(Z(J))$.

$\star$ 7. Zariski topology.

Let M be any set. A *topology* on M is a collection τ of subsets of M such that

(a) $\emptyset \in \tau$;

(b) $M \in \tau$;

(c) τ is closed under arbitrary unions: if $U_\alpha \in \tau$ for α in some index set A, then $\cup_{\alpha \in A} U_\alpha \in \tau$;

(d) τ is closed under finite intersections: if $U_1, \ldots, U_s \in \tau$ for some integer s, then $\cap_{i=1}^{s} U_i \in \tau$.

If τ is a topology on M, then the elements of τ are called the *open sets* of the topology. A subset of M is *closed* if its complement is open.

For example, the usual topology on $\mathbb{R}^n$ is formed by calling a set open if it contains an open ball about each of its points.

(a) Show that $\emptyset$ and $\mathbb{A}^n$ are algebraic sets.

(b) Show that an arbitrary intersection of algebraic sets is an algebraic set.

(c) Show that a finite union of algebraic sets is an algebraic set. (Hint: show $Z(I) \cup Z(J) = Z(\{fg : f \in I,\ g \in J\})$.)

(d) Explain why it follows that the collection of complements of algebraic sets forms a topology on $\mathbb{A}^n$.

(e) The topology $\mathbb{A}^n$ described above is called the *Zariski topology*. It is the usual topology of interest to algebraic geometers. Draw some examples of (Zariski) open sets in $\mathbb{A}^2_{\mathbb{R}}$.

(f) Show that the Zariski topology is not Hausdorff in general. That is, given an example of points $p, q \in \mathbb{A}^n$ such that there are no open sets U containing p and V containing q with $U \cap V = \emptyset$. In other words, we cannot necessarily surround distinct points by disjoint open sets. This is quite a difference with the usual topology on $\mathbb{R}^n$.

8. Show that if $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$, then $\{p\}$ is an algebraic set. Show that finite subsets of $\mathbb{A}^n$ are algebraic sets.

9. Draw pictures of $Z(y - x^2)$ over $\mathbb{C}$ and over $\mathbb{Z}/7\mathbb{Z}$.

10. Calculate $Z(x^2+y^2-1, (x-3)^2+y^2-1) \subset \mathbb{A}^2_k$ for $k = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}/7\mathbb{Z}$ (an intersection of two circles).