PCMI 2008 Undergraduate Summer School

Lecture 15: Sandpiles

David Perkinson

Reed College
Portland, OR

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Beginnings

- *Self-Organized Criticality: An Explanation of* $1/f$ *Noise*, Bak, Tang, Wiesenfeld, Physical Review Letters, 1987.
- *Self-Organized Critical State of Sandpile Automaton Models*, Dhar, Physical Review Letters, 1990.

Abelian Sandpile Model

Notation

$G = (V, E, s)$: finite, connected, loopless, multigraph with a distinguished vertex s called the sink

sandpile configurations: $\mathbb{N}^{V_s}$ where $V_s = V \setminus \{s\}$

monoid structure: $(c + c')_v = c_v + c'_v$ for all $v \in V_s$

Stability

$$\deg(v) = |\{\{v, w\} \in E : w \in V\}|$$

$v \in V_s$ is stable in a configuration c if $c_v < \deg(v)$

Toppling

If v is an unstable vertex in c, we get a new configuration c' by toppling c at v:

$$c'_w = \begin{cases} c_w + 1 & \text{if } \{v, w\} \in E \\ c_w - \deg(w) & \text{if } w = v \\ c_w & \text{otherwise} \end{cases}$$

- By a series of topplings, every configuration reaches a stable configuration.
- This stable configuration is independent of the ordering of the topplings.

Monoid of Stable Sandpiles

Let $\mathcal{S}$ denote the commutative monoid of stable sandpile configurations with addition define by

$$c \circledast c' = \text{stabilization}(c + c').$$

Recurrent Configurations

Definition

A stable configuration c is recurrent if given any configuration c', there exists a configuration c'' such that

$$\text{stabilization}(c' + c'') = c.$$

Example

Define $c_{\max}$ by $(c_{\max})_v = \deg v - 1$. Then $c_{\max}$ is recurrent.

HW

The recurrent configurations are exactly configurations that can be reached by adding sand to $c_{\max}$ and stabilizing.

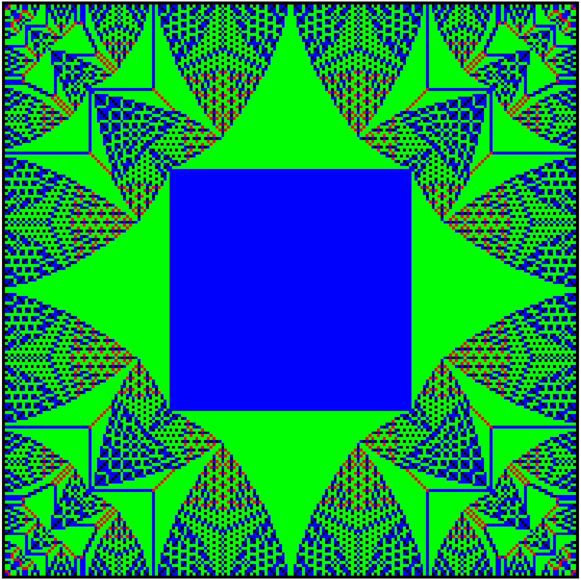
The Sandpile Group

Theorem/Definition

The collection of recurrent configurations, $\mathcal{G}$ is a group called the sandpile group.

Interesting question:

What is the identity of $\mathcal{G}$?



Laplacian

Let $V_s = \{v_1, \ldots, v_n\}$.

Definition

The reduced Laplacian matrix for G is the $n \times n$ matrix, L, where

$$L_{ij} = \begin{cases} \deg v_i & \text{if } i = j \\ -m & \text{if } \{i,j\} \in E,\ m \text{ times} \\ 0 & \text{otherwise} \end{cases}$$

Note

If c is a configuration with unstable vertex v_i, then

$$c - Le_i$$

is the configuration obtained by toppling v_i.

Theorem

$$\mathcal{G} \approx \mathbb{Z}^n / image(L)$$

Corollary

$$|\mathcal{G}| = \det L = \textit{ the number of spanning trees of } G$$

Proof.

Matrix-tree theorem. □

Algebraic geometry of sandpiles

Let $V = \{v_1, \ldots, v_{n+1}\}$ where

- $v_{n+1} = s$, the sink.
- If v_i is farther than v_j from the sink, then $i > j$.

Label v_i with the indeterminate x_i for each i.

For $i \in \{1, \ldots, n+1\}$,

$$p_i = x_i^{\deg v_i} - \prod_{j:\{v_i,v_j\}\in E} x_j, \quad \text{setting } x_{n+1} = 1.$$

Definition

The sandpile ideal for G is

$$I_G = (p_i : i = 1, \ldots, n+1) \subseteq \mathbb{C}[x_1, \ldots, x_n]$$

Consider $\mathbb{C}[x_1, \ldots, x_n]$ with graded revlex monomial ordering and

$$x_1 > \cdots > x_n.$$

Theorem (Cori, Rossin, Salvy, 2006)

A normal basis for $\mathbb{C}[x_1, \ldots, x_n]/I_G$ with respect to the above ordering is in one-to-one correspondence with the elements of the sandpile group, $\mathcal{G}$:

$$x^e \quad \longleftrightarrow \quad c_{\max} - e.$$

Theorem (P., 2008)

$Z(I_G) \subset \mathbb{A}^n$ is a set of $|\mathcal{G}|$ points forming an orbit of a representation of $\mathcal{G}$ by a group of $n \times n$ matrices.

Theorem (Baker, et al., 2007)

There is more than an analogy between algebraic curves and graphs: Riemann-Roch, Riemann-Hurwitz, Jacobi inversion, etc. The sandpile group plays the role of the Picard group.

Corollary

The postulation number for the (homgenization of the) sandpile ideal is the genus of the graph: $g = |E| - |V| + 1$.

Tilings

Order of the all-2s sandpile on an even square grid

Let g_n be the order of the all-2s element of a $2n \times 2n$ grid.

The first few values of g_n are

$$g_n : 1, 3, 29, 901, 89893$$

Tilings of the even square grid

Let T_n be the number of domino tilings of a $2n \times 2n$ grid.

$$T_n = 4^{n^2} \prod_{i,j=1}^{n} \left(\cos^2 \frac{i\pi}{2n+1} + \cos^2 \frac{j\pi}{2n+1} \right)$$
$$= 2^n a_n^2$$

where the first few values of a_n are

$$a_n : 1, 3, 29, 901, 89893$$

A few more values of a_n and g_n

a_n	1	3	29	901	89,893	28,793,575	29,607,089,625
g_n	1	3	29	901	89,893	5,758,715	22,687,425

Theorem (Morar, P., 2007)

For each n, there are graphs P_n and A_n such that

- *The number of domino tilings (perfect matchings) of A_n is a_n.*
- *The spanning trees of P_n are in 1-1 correspondence with the domino tilings of A_n.*
- *The cyclic subgroup generated by the all-2s element injects into the sandpile group of P_n.*

Thus, g_n divides a_n.

Theorem (Morar, P., 2007)

Let F_i be the i-th Fibonnaci number (starting with $F_1 = F_2 = 1$). The order of the all-2s element for the $2 \times n$ grid is:

1. *for $n = 2m$,*

$$\mathrm{order}(\textit{all 2s}) = \begin{cases} F_{n+1}/2 & \text{if } 3|(m-1) \\ F_{n+1} & \text{otherwise.} \end{cases}$$

2. *for $n = 2m - 1$,*

$$\mathrm{order}(\textit{all-2s}) = \begin{cases} (F_n + F_{n+2})/2 & \text{if } 3|m \\ F_n + F_{n+2} & \text{otherwise.} \end{cases}$$

In particular, the order of the all-2s element is odd.

Complexity

Sandpile as a universal computer, Goles and Margenstern, International Journal of Modern Physics, 1996.

Constructing a sandpile machine, Schoenberg-Jones, Reed College Thesis, 2008.