PCMI 2008 Undergraduate Summer School

Lecture 12: Schubert Calculus II

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Review

- cycles

$$\sum n_i V_i \in Z_r(X)$$

- rational equivalence

$$V \sim W$$

- codimension r cycles modulo rational equivalence

$$A^r(X) = Z_{n-r}(X)/\sim$$

- Chow ring

$$A^*(X) = \oplus_{r=0}^{n} A^r(X)$$

$$[V] \cdot [W] = [V \cap W]$$

- flag

$$A_0 \subsetneq \cdots \subsetneq A_r$$

- Schubert variety

$$\mathfrak{S}(A_0, \ldots, A_r) = \{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}$$

- Schubert class

$$(a_0, \ldots, a_r) = [\mathfrak{S}(A_0, \ldots, A_r)] \in A^*(\mathbb{G}_r\mathbb{P}^n)$$

  - depends only on the dimensions, $a_i = \dim A_i$.
  - $\operatorname{codim}(a_0, \ldots, a_r) = (r+1)(n-r) - \sum_{i=0}^{r}(a_i - i)$.
  - $A^*(\mathbb{G}_r\mathbb{P}^n)$ is free abelian on the Schubert classes.

Example

$\mathbb{G}_1\mathbb{P}^3$ dimension $= 4$, $0 \le a_0 < a_1 \le 3$.

Schubert classes

codimension	class	condition
0	$(2,3)$	no condition
1	$(1,3)$	meet a line
2	$(0,3)$	pass through a point
2	$(1,2)$	lie in a plane
3	$(0,2)$	pass through a point and lie in a plane
4	$(0,1)$	be a certain line

Goal

Describe multiplication in $A^*(\mathbb{G}_r\mathbb{P}^n)$.

Change of notation

Given the class $(a_0, \dots, a_r)$, let $\lambda_i = n - r - (a_i - i)$ for all i.

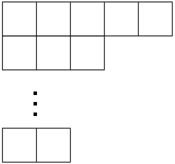
We write $\{\lambda_0, \dots, \lambda_r\} = (a_0, \dots, a_r) \in A^*(\mathbb{G}_r\mathbb{P}^n)$.

- $n - r \geq \lambda_0 \geq \cdots \geq \lambda_r \geq 0$
- $\text{codim}\{\lambda_0, \dots, \lambda_r\} = |\lambda| = \sum_i \lambda_i$

codim	class	condition
0	$(2,3)$, $\{0,0\}$	no condition
1	$(1,3)$, $\{1,0\}$	meet a line
2	$(0,3)$, $\{2,0\}$	pass through a point
2	$(1,2)$, $\{1,1\}$	lie in a plane
3	$(0,2)$, $\{2,1\}$	pass through a point and lie in a plane
4	$(0,1)$, $\{2,2\}$	be a certain line

Young diagrams

For each λ: $\lambda_0 \geq \cdots \geq \lambda_r \geq 0$, there is an associated Young diagram



λ_0 boxes

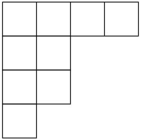
λ_1 boxes

$\vdots$

λ_r boxes

Example

$\{4, 2, 2, 1\}$



Multiplication

Theorem

For $\{\lambda\}, \{\mu\} \in A^*(\mathbb{G}_r\mathbb{P}^n)$,

$$\{\lambda\} \cdot \{\mu\} = \sum_{\nu} N_{\lambda\mu\nu}\{\nu\}$$

the sum over $\{\nu\} \in A^*(\mathbb{G}_r\mathbb{P}^n)$.

The Littlewood-Richardson number, $N_{\lambda\mu\nu}$, is the number of *strict* μ*-expansions of* λ *giving* ν*.*

Definition

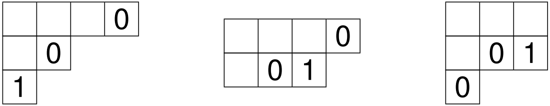
A μ-expansion of λ is a Young diagram obtained inductively. Start with the Young diagram for λ. At the i-step:

- Add μ_i boxes (to existing rows or to the bottom).
- No two added boxes allowed in the same column.
- The result must be a valid Young diagram.
- Write the number i in each of the boxes.

A μ-expansion is strict if when reading off the integers in the boxes right-to-left, starting with the top row and working down, at no time is a number read more times than a smaller number has already been read.

Example

Some $\{2, 1\}$-expansions of $\{3, 1\}$



strict strict not strict

Multiplication table for $A^*(\mathbb{G}_1\mathbb{P}^3)$

codim	$*$	$\circ$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{2}$	$\ydiagram{2,1}$	$\ydiagram{2,2}$
0	$\circ$						
1	$\ydiagram{1}$						
2	$\ydiagram{1,1}$						
2	$\ydiagram{2}$						
3	$\ydiagram{2,1}$						
4	$\ydiagram{2,2}$						

Tips for calculating products

$$\{\lambda\} \cdot \{\mu\} = \sum_{\nu} N_{\lambda\mu\nu}\{\nu\}$$

- If codim $\{\lambda\}$ + codim $\{\mu\} > \dim \mathbb{G}_r\mathbb{P}^n$, then $\{\lambda\} \cdot \{\mu\} = 0$.
- $\{\nu\} = 0$ if $\nu_0 > n - r$ or the number of rows in its Young diagram is greater than $r + 1$.
- If the Young diagram for $\{\nu\}$ has fewer than r rows, remember to pad $\{\nu\}$ with trailing 0s to express its class in the Chow ring.