PCMI 2008 Undergraduate Summer School

Lecture 11: Schubert Calculus I

David Perkinson

Reed College
Portland, OR

Summer 2008

Question

How many lines meet for general lines L_1, L_2, L_3, L_4 in $\mathbb{R}^3$?

Answer 1: Consider the surface of lines meeting L_1, L_2, L_3.

- Go to each point on L_1 and draw a line through the point where L_2 and L_3 appear to meet.
- The resulting collection of lines is a quadric surface.
- Intersecting that surface with L_4 gives 2 points.
- These two points correspond to the 2 solutions.

Exercise

Show that each point on the saddle surface, $z = xy$, is contained in exactly two lines lying on the surface.

Answer 2: Specialize.

Suppose L_1 meets L_2 and L_3 meets L_4.

- One solution is the line through the points $L_1 \cap L_2$ and $L_3 \cap L_4$.
- The other solution is the line of intersection between the two planes spanned by L_1, L_2 and L_3, L_4.

Answer 3: Intersection theory.

$$\{\text{lines meeting } L_i\} = \mathbb{G}_1\mathbb{P}^3 \cap H_i$$

for some hyperplane $H_i \subset \mathbb{P}^5$.

$$\begin{aligned}\{\text{lines meeting } L_1, L_2, L_3, L_4\} &= \cap_i \left(\mathbb{G}_1\mathbb{P}^3 \cap H_i\right) \\ &= \mathbb{G}_1\mathbb{P}^3 \cap (\cap_i H_i)\end{aligned}$$

How many times does the line $\cap_i H_i$ meet $\mathbb{G}_1\mathbb{P}^3 \subset \mathbb{P}^5$?

Answer: 2 (by Bezout).
More simply, parametrize the line:

$$t \mapsto (a_0 t + b_0, \ldots, a_5 t + b_5)$$

then plug it in to the equation $x_0x_5 - x_1x_4 + x_2x_3 = 0$ defining $\mathbb{G}_1\mathbb{P}^3 \subset \mathbb{P}^5$. Solve the resulting quadratic equation in t.

Goal

Generalize these arguments.

The Chow Ring

X a variety of dimension n.

Definition

An r-cycle is a finite formal sum, $\sum_i n_i V_i$ where each $n_i \in \mathbb{Z}$ and each V_i is an r-dimensional subvariety of X.

Notation:

$$Z_r(X) = \{\text{all } r\text{-cycles of } X\}$$

Definition

V has codimension r in X if $\dim V = n - r$.

Definition

Subvarieties $V, W \subseteq X$ of dimension r are rationally equivalent if W is a continuous deformation of V.

Notation:

$$V \sim W$$

Definition

$$A^r(X) = Z_{n-r}(X)/\sim$$

The Chow ring of X is

$$A^*(X) = \oplus_{i=0}^{n} A^r(X).$$

Ring structure on $A^*(X)$

Definition

For $[V] \in A^r(X)$ and $[W] \in A^s(X)$, define

$$[V] \cdot [W] = [V \cap W] \in A^{r+s}(X)$$

after deforming V and W so that they meet *transversally*.

Definition

V and W meet transversally at $p \in V \cap W$ if the tangent spaces for V and W at p together span the tangent space of X.

V and W meet transversally if they meet transversally at each point $p \in V \cap W$.

Example

$$A^*(\mathbb{P}^n) \approx \mathbb{Z}[t]/(t^{n+1})$$
$$[V] \mapsto \deg(V)\, t^{\operatorname{codim}(V)}$$

n=2

p, q points in $\mathbb{P}^2$, $X = Z(yz - x^2)$, $Y = Z(zy^2 - x^3 - zx^2)$

- $2[p] + 3[q] - [X] + 5[Y] + 4[\mathbb{P}^2] \mapsto 2t^2 + 3t^2 - 2t + 15t + 4 = 5t^2 + 13t + 4.$

- $[X] \cdot [Y] \mapsto (2t)(3t) = 6t^2$ (X, Y meet in 6 points).
- $(2[p] + [X] + [\mathbb{P}^2])^2 \mapsto (2t^2 + 2t + 1)^2 = 8t^2 + 4t + 1.$

Goal

Describe the Chow ring $A^*(\mathbb{G}_r\mathbb{P}^n)$.

Note: $A^r(\mathbb{G}_r\mathbb{P}^n) = H^{2r}(\mathbb{G}_r\mathbb{P}^n, \mathbb{Z})$.

Schubert Varieties

Definition

A sequence

$$A_0 \subsetneq \cdots \subsetneq A_r$$

where each A_i is a linear subspace of $\mathbb{P}^n$ is called a flag.

Definition

Fixing a flag as above, define the corresponding Schubert variety by

$$\mathfrak{S}(A_0, \ldots, A_r) = \{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}$$

Proposition

$$\mathfrak{S}(A_0, \ldots, A_r) = \mathbb{G}_r\mathbb{P}^n \cap M$$

for some linear subspace $M \subset \mathbb{P}^N$*.*

M is a hyperplane iff $\dim A_0 = n - r - 1$ *and* $\dim A_i = n - r + i$ *for* $i = 1, \ldots, r$*.*

Proposition

If $A_0 \subsetneq \cdots \subsetneq A_r$ *and* $B_0 \subsetneq \cdots \subsetneq B_r$ *are flags with* $\dim A_i = \dim B_i$ *for all* i*, then*

$$[\mathfrak{S}(A_0, \ldots, A_r)] = [\mathfrak{S}(B_0, \ldots, B_r)] \in A^*(\mathbb{G}_r\mathbb{P}^n).$$

Notation

Letting $a_i = \dim A_i$, we write

$$\mathfrak{S}(a_0, \ldots, a_r) \qquad \text{or} \qquad (a_0, \ldots, a_r)$$

for the cycle class $[\mathfrak{S}(A_0, \ldots, A_r)]$.

Theorem

$A^*(\mathbb{G}_r\mathbb{P}^n)$ is a free abelian group on

$$\{(a_0, \ldots, a_r) : 0 \le a_0 < \cdots < a_r \le n\}.$$

$(a_0, \ldots, a_r) \in A^\ell(\mathbb{G}_r\mathbb{P}^n)$ where $\ell = (r+1)(n-r) - \sum_{i=0}^{r}(a_i - i)$.

Next time

Describe the multiplication in $A^*(\mathbb{G}_r\mathbb{P}^n)$.