Lotsa Dots:
Self-Dual Affine Monomial Varieties

A Thesis
Presented to
The Division of Mathematics and Natural Sciences
Reed College

In Partial Fulfillment
of the Requirements for the Degree
Bachelor of Arts

John C. Mulliken

May 1995

Approved for the Division
(Mathematics)

David Perkinson

Acknowledgements

To my parents, Richard and Susan Mulliken, for tolerating, supporting, and even encouraging my many crazy dreams, and to my advisors, David Perkinson and Jerry Shurman, who taught me almost everything I know about mathematics, and who were thoughtful and funny along the way.

Thanks to Zinovy Reichstein for providing the result I needed to see that there actually were lots of examples of self-dual affine monomial varieties out there. Thanks also to all of my friends who helped me immensely just by being there.

Table of Contents

Abstract

This thesis is concerned with self-dual affine monomial varieties. Duality is defined in terms of higher-order tangent spaces and generalizes the classical notion of duality in the familiar sense of a plane curve. The main result is a characterization of self-duality in terms of Hilbert functions. The existence of an infinite class of self-dual surfaces is shown.

Chapter 1

Introduction

Affine monomial varieties are of interest because of the peculiar relationship between their geometric properties and the combinatorial properties of their "exponent sets." They also form the building blocks for more complex geometric objects called toric varieties. The study of the higher-order osculating spaces of these varieties leads to the concept of self-duality which is the focus of this thesis.

In Chapter 1, the necessary concepts and tools are introduced, a method for determining whether a given affine monomial variety is self-dual is presented, and several examples are explored. Chapter 2 gives the results of this thesis along with their proofs. Chapter 3 consists of a listing of all currently known self-dual affine monomial varieties.

1.1 Duality of Plane Curves

The construction of duals to plane curves is relatively simple, and exhibits the geometric concepts used in this thesis in an elementary setting. The process by which we construct the duals of affine monomial varieties is entirely analagous, so the reader not familiar with the concepts of algebraic geometry may wish to get accustomed to the procedure.

Consider a parametrized plane curve such as the parabola

$$t \mapsto (t, t^2).$$

This is a parametrization of the graph of the well-known parabola $y = x^2$. At each point of the parabola (equivalently for each value of t) we can find the tangent line. As we know, we can completely describe any line by its y-intercept and its slope. Hence we can form a one-to-one correspondence between points in $\mathbb{R}^2$ and lines in $\mathbb{R}^2$, identifying

$$\text{the line } y = mx + b \leftrightarrow \text{the point } (b, m).$$

If we examine the parabola, we find that the tangent line at time t is given by $y = 2tx - t^2$. This line can be represented in the "dual-space" consisting of parameter points by the point $(-t^2, 2t)$. At time $t = 2$, the tangent line is $y = 4x-4$, as pictured in Figure 1.1. The point in the dual space corresponding to this tangent

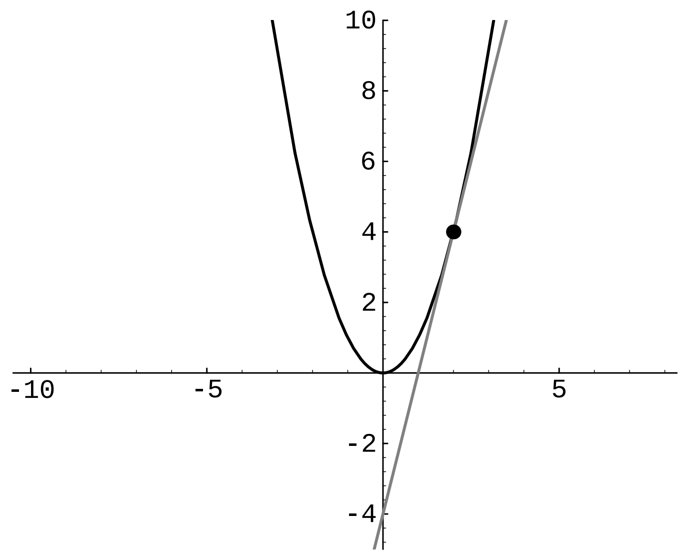


Figure 1.1: The parabola (t, t^2) and the tangent line at $t = 2$

line is $(-4, 4)$, The set of all points corresponding to tangent lines as t moves along the real number-line is another curve in $\mathbb{R}^2$ parametrized by $(-t^2, 2t)$. We note that this is also a parabola, parametrizing the graph of $y^2 = -4x$.

In general, we say that the **dual curve** of a plane curve is the set of points parametrizing the set of all tangent lines to the curve. A plane curve is **self-dual** if we can find a change of coordinates taking the parametrization of the dual to the parametrization of the original curve. For example we have just seen that the parabola (t, t^2) is self-dual since the only difference between the curve and its dual is a reversal of the order of the components and multiplying them by 2 and -1 respectively. In fact, the graph of any polynomial $y = f(x)$ can be parametrized as

$$t \mapsto (t, f(t)).$$

With a bit of calculation, it is not difficult to see that the only self-dual curve parametrized in the form $(t, f(t))$ is the parabola.

As a final example, consider the nodal cubic $y^2 = x^2(x + 1)$ which can be parametrized as $(t^2 - 1, t^3 - t)$. The tangent line at time t is given by

$$y = \frac{3t^2 - 1}{2t} x - \frac{(t-1)^2(t+1)^2}{2t}.$$

Thus the dual-curve to the nodal cubic can be parametrized as

$$t \mapsto \left(-\frac{(t-1)^2(t+1)^2}{2t}, \frac{3t^2 - 1}{2t}\right).$$

The nodal cubic is shown along with its dual curve in Figure 1.2. It's immediately apparent that this curve is not self-dual.

It turns out that the dual of the dual of a plane curve (with the exception of lines) is in fact the original curve, as can be easily verified for the given examples.

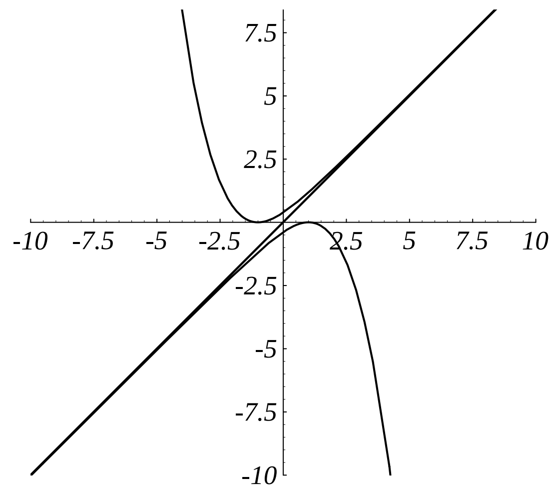


Figure 1.2: The nodal cubic and its dual curve

In higher dimensions, we will see that certain duals may not exist, and in the case where they do, it's possible that the dual of the dual may not be the original curve.

1.2 The Varieties Themselves

Though the objects which we will be studying have a rather simple definition, their geometry is relatively complicated. The development of the theory will be much simpler to follow if the reader refers to the intuitions just developed based on the duals of plane curves. The question of which manifolds are self-dual is intractable in general. The tools of algebraic geometry are useful here because of the essential simplicity of the manifolds or varieties we are studying.

Put simply, affine monomial varieties are mappings of complex affine spaces, whose component functions are monomials.

Definition 1.2.1. *An* **affine monomial variety** v *is a mapping*

$$\begin{aligned} v : \mathbb{C}^n &\rightarrow \mathbb{C}^{t+1} \\ x &\mapsto (a_0 x^{m_0}, a_1 x^{m_1}, \ldots, a_t x^{m_t}) \end{aligned}$$

where $x^{m_i} = \prod_{j=1}^n x^{m_ij}$ *for* $m_{ij} = (m_{i1}, \ldots m_{in}) \in \mathbb{Z}^n$ *and* $a_i \in \mathbb{C}$ *for* $0 \leq i \leq t$. *We consider two varieties the same if they are equivalent under a "toric change of coordinates", i.e. if they have identical components up to scaling by non-zero coefficients and reordering.*

An affine monomial variety is **standard** *if* $m_0 = 0$, $a_i = 1$ *for* $0 \leq i \leq t$, *and* $m_i \leq m_{i+1}$ *for* $0 \leq i < t$ *in the standard ascending lexicographic ordering.*

We force the leading coordinate of standard mappings to be one because it will make our final results easier to state and prove. The "extra coordinate" may be considered simply as a placeholder, or the means by which we will eventually homogenize the coordinates of the mapping.

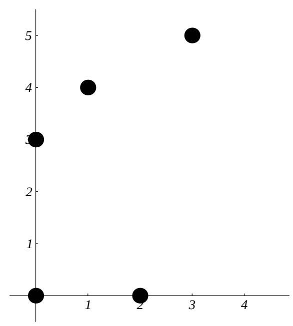


Figure 1.3: The lattice points in V_1

Two examples of affine monomial varieties are:

$$\begin{array}{rcl} v_1: \quad (x,y) & \mapsto & (1, x^2, y^3, xy^4, x^3y^5) \\ v_2: \quad (x,y,z) & \mapsto & (1, x, y, z, xy, xz, yz, xyz) \end{array}$$

Associated with each affine monomial variety is a set of lattice points: the exponents of the monomials.

Definition 1.2.2. *To each affine monomial variety v we associate a* **lattice set** $V \subset \mathbb{Z}^n_{\geq 0}$, *given by*

$$V = \{m_0, m_1, \ldots, m_t\}.$$

For example, the two affine monomial varieties shown above correspond to the lattice sets:

$$\begin{array}{rcl} V_1 & = & \{(0,0),(2,0),(0,3),(1,4),(3,5)\} \\ V_2 & = & \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\} \end{array}$$

The position of the points of V_1 are shown in Figure 1.3. The points in V_2 are the vertices of the unit cube.

1.3 The Varieties and their Tangent Spaces

To define the dual of a plane curve, we think of a tangent line as corresponding to a point in a dual plane, as described in Section 1.1. To develop the notion of a dual to one of our varieties, we need to examine a more general concept of tangent spaces.

Definition 1.3.1. *The k-th order* **osculating space** *of an affine monomial variety v is given by*

$$\mathrm{Osc}_k v = \mathrm{Span}\{v_a\}_{0 \leq |a| \leq k},$$

where $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, $|a| = \sum a_i$, *and* $v_a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}$.

As a convenient technical device, we place the partial derivatives in a matrix. We will see that this use of the language of linear algebra simplifies matters a great deal.

Definition 1.3.2. *The* **matrix of k-jets** *of an affine monomial variety v is the matrix consisting of the partial derivatives of v up to order k as rows:*

$$J_k v = [v_a]_{0 \leq |a| \leq k}.$$

First, we note that for $x \in \mathbb{C}^n$ the rank of $J_k v(x)$ is $\dim(\mathrm{Osc}_k v(x)) + 1$. Since the rank can be calculated as the number of certain minor determinants which vanish, the rank will have a generic value d_k. Points where $\mathrm{rank}(J_k v) < d_k$ are called **k-th order inflections.** The inflectionary behavior of affine monomial varieties is explored in [3] and [6].

The dimension of the space spanned by the first partials shows how much the surface is flexing to first order, while the dimension of the space spanned by the partials up to degree two shows how much the tangent space is flexing, and so on. The rank of the matrix of k-jets measures how much the $(k-1)$-th osculating space is "flexing" or "moving around" in its ambient space. For a perfectly uninflected variety, the rank of the matrix of k-jets would be $\binom{n+k}{n}$. The amount that the generic dimension drops below this is a measure of how "lazy" the surface is being in general. At a k-th order inflection, the surface is being even lazier than at a generic point, and more of its partial derivatives are vanishing.

1.4 The Varieties and their Dual-Spaces

In imitation of the case of plane curves, we would naively like to consider the dual to be the set of all hyperplanes tangent to the variety v at a given point. Unfortunately, there will often be more than one, i.e. there will not be a single, well-defined normal direction. Thus we increase the order of the osculating space under consideration hoping to find one normal direction for each point (except at order k inflections). We will see later that we can use the language of linear algebra to simplify this process. We thus make the following definition.

Definition 1.4.1. *The k-th order* **dual** *of v is*

$$\mathrm{Dual}_k v = \left\{ \text{hyperplanes } H \subset \mathbb{P}^{t*} : \begin{array}{l} H \supset \mathrm{Osc}_k v(p) \\ \text{for some } p \in \mathbb{C}^n \text{ such that} \\ \mathrm{rank}(J_k v(p)) = d_k \end{array} \right\}^{-}.$$

We note for those not familiar with the concept of projective space that $\mathbb{P}^{t*}$ is $\mathbb{C}^{t+1}$ where we identify points that are non-zero scalar multiples of one another. We think of the hyperplane with defining equation $\sum_{i=0}^{t} a_j x_i = 0$ as the point $(a_0, a_1, \ldots, a_t) \in \mathbb{P}^{t*}$. We note in passing that scaling all the coefficients a_i by a non-zero scalar does not change the hyperplane. The closure in the above definition is in the algebreo-geometric sense of the smallest set defined by homogenous polynomial equations vanishing on the above set. These concepts are fully explained in any basic text on algebraic geometry, for instance [1] or [4].

When the dimension of $\mathrm{Dual}_k v$ is 1, there is a unique hyperplane containing the k-th order osculating space at all non-inflected points. It is not hard to show that, in this case, $\mathrm{Dual}_k v$ will be the projective closure of a *monomial* variety. For details see the beginning of Chapter 2. We call this affine monomial variety the **dual variety** of the original variety, and note that this dual will not exist for many affine monomial varieties. From this point on, we blur the distinction between the dual variety and the dual, using the notation $\mathrm{Dual}_k v$ to refer to the dual variety, if it exists.

We choose the following definition of self-duality, which generalizes the classical notion of self-duality from plane curves.

Definition 1.4.2. *An affine monomial variety v is* **self-dual** *if it is equal to its dual, up to a toric change of coordinates, i.e. if they have identical components up to scaling by non-zero coefficients and reordering.*

Recall that we consider two monomial varieties to be the same if they are the same up to a toric change of coordinates. Thus we can reorder the component monomials and multiply them by any non-zero constants without changing the affine monomial variety. This condition is equivalent to saying that two varieties are the same if they have the same corresponding lattice sets.

For instance, the following two varieties are considered identical.

$$(1, x, y, xy^2) \leftrightarrow (-x, 2xy^2, \frac{15}{3}, -12y)$$

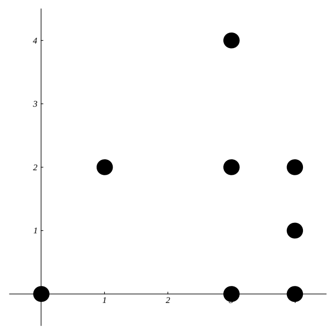


Figure 1.4: Points Chosen at Random

1.5 Examples

All the examples we consider will be surfaces, because we can represent their lattice sets graphically and can easily see the workings of the theory in this context. All higher-dimensional varieties can be analyzed using the same method.

Since every element of the kernel of $J_k v$ will dot to zero with each partial derivative of v of order $\leq k$, each element of a basis for $\ker(J_k v)$ will be a direction perpendicular to the osculating space $\mathrm{Osc}_k v$. Thus when we find a k such that $\dim(\ker(J_k v)) = 1$, we have found a unique hyperplane containing all the osculating spaces of degree $\leq k$. The coefficients of this hyperplane's defining equation will be the component functions of the dual-variety to v.

1.5.1 The Generic Case: Points Chosen at Random

To begin with, we choose a set of points in the quarter-plane at random, including the origin as we must for any standard affine monomial variety. Here we examine the variety corresponding to

$$V = \{(0,0),(3,0),(4,0),(4,1),(1,2),(3,2),(4,2),(3,4)\},$$

pictured in Figure 1.4.

The affine monomial variety corresponding to v, as well as the various matrices of k-jets and their respective kernels, are reproduced in Table 1.1. In the middle column we see the matrix of k-jets, and in the third column the matrix whose columns span the kernel of the corresponding matrix of k-Jets. We observe that the dimension of these kernels drops with each increase in k, and becomes constant at 0 for $k \geq 3$. We note in particular that the dimension of the kernel of the matrix

Table 1.1: Points Chosen at Random

$$v(x,y) = (1, x^3, x^4, x^4y, xy^2, x^3y^2, x^4y^2, x^3y^4)$$

k	$J_k(v)$	$\ker(J_k(v))$
0	$\begin{bmatrix} 1 & x^3 & x^4 & x^4y & xy^2 & x^3y^2 & x^4y^2 & x^3y^4 \end{bmatrix}$	$\begin{bmatrix} -x^3 & -xy^2 & -x^4 & -x^4y & -x^3y^2 & -x^4y^2 & -x^3y^4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
1	$\begin{bmatrix} 1 & x^3 & x^4 & x^4y & xy^2 & x^3y^2 & x^4y^2 & x^3y^4 \\ 0 & 3x^2 & 4x^3 & 4x^3y & y^2 & 3x^2y^2 & 4x^3y^2 & 3x^2y^4 \\ 0 & 0 & 0 & x^4 & 2xy & 2x^3y & 2x^4y & 4x^3y^3 \end{bmatrix}$	$\begin{bmatrix} x^4 & 2x^3y^2 & -4x^4y^2 & -x^4y^2 & 4x^3y^4 \\ -4x & -2y^2 & 7xy^2 & 4xy^2 & -y^4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6y & -6y & 0 \\ 0 & -3x^2 & 3x^3 & 0 & -6x^2y^2 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$
2	$\begin{bmatrix} 1 & x^3 & x^4 & x^4y & xy^2 & x^3y^2 & x^4y^2 & x^3y^4 \\ 0 & 3x^2 & 4x^3 & 4x^3y & y^2 & 3x^2y^2 & 4x^3y^2 & 3x^2y^4 \\ 0 & 0 & 0 & x^4 & 2xy & 2x^3y & 2x^4y & 4x^3y^3 \\ 0 & 6x & 12x^2 & 12x^2y & 0 & 6xy^2 & 12x^2y^2 & 6xy^4 \\ 0 & 0 & 0 & 4x^3 & 2y & 6x^2y & 8x^3y & 12x^2y^3 \\ 0 & 0 & 0 & 0 & 2x & 2x^3 & 2x^4 & 12x^3y^2 \end{bmatrix}$	$\begin{bmatrix} -x^4y^2 & -x^4y^4 \\ 4xy^2 & 5xy^4 \\ -3y^2 & -7y^4 \\ 0 & 8y^3 \\ 2x^3 & 2x^3y^2 \\ -6x & -8xy^2 \\ 4 & 0 \\ 0 & x \end{bmatrix}$
3	$\begin{bmatrix} 1 & x^3 & x^4 & x^4y & xy^2 & x^3y^2 & x^4y^2 & x^3y^4 \\ 0 & 3x^2 & 4x^3 & 4x^3y & y^2 & 3x^2y^2 & 4x^3y^2 & 3x^2y^4 \\ 0 & 0 & 0 & x^4 & 2xy & 2x^3y & 2x^4y & 4x^3y^3 \\ 0 & 6x & 12x^2 & 12x^2y & 0 & 6xy^2 & 12x^2y^2 & 6xy^4 \\ 0 & 0 & 0 & 4x^3 & 2y & 6x^2y & 8x^3y & 12x^2y^3 \\ 0 & 0 & 0 & 0 & 2x & 2x^3 & 2x^4 & 12x^3y^2 \\ 0 & 6 & 24x & 24xy & 0 & 6y^2 & 24xy^2 & 6y^4 \\ 0 & 0 & 0 & 12x^2 & 0 & 12xy & 24x^2y & 24xy^3 \\ 0 & 0 & 0 & 0 & 2 & 6x^2 & 8x^3 & 36x^2y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24x^3y \end{bmatrix}$	$\vec{0}$

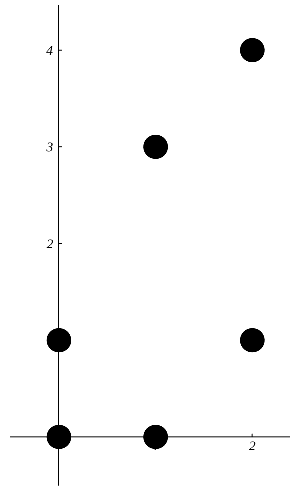


Figure 1.5: The Octagon

of k-Jets never reaches 1, so this variety cannot be self-dual– it doesn't even have a dual. This isn't very surprising for a set of exponents chosen at random.

1.5.2 Central Symmetry: The Octagon

The first theorem we will prove in the results section is that if an affine monomial variety is self-dual then it has a centrally-symmetric corresponding lattice set. Central symmetry is the property that the lattice set has a midpoint (equivalent to the "center of mass" of the points) such that a given point is in the lattice set if and only if its reflection through the midpoint is also in the set. Note that this implies two interesting facts about self-dual varieties: firstly that there can be an odd number of points only if the midpoint is in the lattice set, and secondly that twice the midpoint is always in the set, since the origin must be. This surprising symmetry property suggests the existence of several self-dual affine monomial varieties such as the following one.

The variety corresponding to the lattice set

$$V = \{(0,0),(1,0),(0,1),(2,1),(1,3),(3,3),(2,4),(3,4)\},$$

pictured in Figure 1.5 will be called The Octagon.

As we can see by examining the matrices of k-jets shown in Table 1.2, the nullity of the matrix of 3-jets of v is 1, and the components are indeed the same as those of the original mapping up to multiplication of some of the components by -1 and the fact that they appear in reversed order. The reversal of order of the component functions is not accidental, and will be important in the proof of the fact that all self-dual affine monomial varieties have centrally-symmetric lattice sets.

It is also interesting, and thus far unexplained, that there is a strong symmetry of the matrices in Table 1.2: for $0 \leq k \leq 3$, $J_k v(x) = \ker(J_{3-k} v(x))$ up to row operations on $J_k v$ or column operations on $\ker(J_k v)$.

Another non-obvious fact, which is worth noting now as it will be important for the solution of the question at hand, is that the dimension of the matrix of k-Jets

Table 1.2: The Octagon

$$v(x,y) = (1, x, y, x^2y, xy^3, x^3y^3, x^2y^4, x^3y^4)$$

k	$J_k(v)$	$\ker(J_k(v))$
0	$\begin{bmatrix} 1 & x & y & x^2y & xy^3 & x^3y^3 & x^2y^4 & x^3y^4 \end{bmatrix}$	$\begin{bmatrix} -x & -y & -x^2y & -xy^3 & -x^3y^3 & -x^2y^4 & -x^3y^4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
1	$\begin{bmatrix} 1 & x & y & x^2y & xy^3 & x^3y^3 & x^2y^4 & x^3y^4 \\ 0 & 1 & 0 & 2xy & y^3 & 3x^2y^3 & 2xy^4 & 3x^2y^4 \\ 0 & 0 & 1 & x^2 & 3xy^2 & 3x^3y^2 & 4x^2y^3 & 4x^3y^3 \end{bmatrix}$	$\begin{bmatrix} 2x^2y & 3xy^3 & 5x^3y^3 & 5x^2y^4 & 6x^3y^4 \\ -2xy & -y^3 & -3x^2y^3 & -2xy^4 & -3x^2y^4 \\ -x^2 & -3xy^2 & -3x^3y^2 & -4x^2y^3 & -4x^3y^3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
2	$\begin{bmatrix} 1 & x & y & x^2y & xy^3 & x^3y^3 & x^2y^4 & x^3y^4 \\ 0 & 1 & 0 & 2xy & y^3 & 3x^2y^3 & 2xy^4 & 3x^2y^4 \\ 0 & 0 & 1 & x^2 & 3xy^2 & 3x^3y^2 & 4x^2y^3 & 4x^3y^3 \\ 0 & 0 & 0 & 2y & 0 & 6xy^3 & 2y^4 & 6xy^4 \\ 0 & 0 & 0 & 2x & 3y^2 & 9x^2y^2 & 8xy^3 & 12x^2y^3 \\ 0 & 0 & 0 & 0 & 6xy & 6x^3y & 12x^2y^2 & 12x^3y^2 \end{bmatrix}$	$\begin{bmatrix} -4x^3y^3 & -3x^2y^4 & -6x^3y^4 \\ 4x^2y^3 & 2xy^4 & 5x^2y^4 \\ 3x^3y^2 & 3x^2y^3 & 5x^3y^3 \\ -3xy^2 & -y^3 & -3xy^3 \\ -x^2 & -2xy & -2x^2y \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3	$\begin{bmatrix} 1 & x & y & x^2y & xy^3 & x^3y^3 & x^2y^4 & x^3y^4 \\ 0 & 1 & 0 & 2xy & y^3 & 3x^2y^3 & 2xy^4 & 3x^2y^4 \\ 0 & 0 & 1 & x^2 & 3xy^2 & 3x^3y^2 & 4x^2y^3 & 4x^3y^3 \\ 0 & 0 & 0 & 2y & 0 & 6xy^3 & 2y^4 & 6xy^4 \\ 0 & 0 & 0 & 2x & 3y^2 & 9x^2y^2 & 8xy^3 & 12x^2y^3 \\ 0 & 0 & 0 & 0 & 6xy & 6x^3y & 12x^2y^2 & 12x^3y^2 \\ 0 & 0 & 0 & 0 & 0 & 6y^3 & 0 & 6y^4 \\ 0 & 0 & 0 & 2 & 0 & 18xy^2 & 8y^3 & 24xy^3 \\ 0 & 0 & 0 & 0 & 6y & 18x^2y & 24xy^2 & 36x^2y^2 \\ 0 & 0 & 0 & 0 & 6x & 6x^3 & 24x^2y & 24x^3y \end{bmatrix}$	$\begin{bmatrix} x^3y^4 \\ -x^2y^4 \\ -x^3y^3 \\ xy^3 \\ x^2y \\ -y \\ -x \\ 1 \end{bmatrix}$

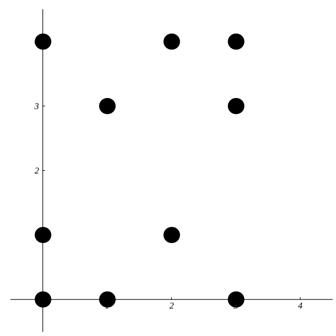


Figure 1.6: The Octagon Plus Two Points

will be strictly decreasing with k until some value of k after which the dimension will be constant at zero[1]. Specifically note that the nullity of the matrix of k-jets cannot remain constant at a non-zero value as k increases—so we can find the value of k for which the dimension of the kernel (and thus the dimension of the linear space of hyperplanes containing the osculating space) is 0. We then examine $\ker(J_{k-1}v)$ and compare it with the original variety to check if the variety is self-dual. If its dimension is not 1, then it definitely isn't self-dual, in fact we can't even make such a comparison as there is be no dual in this case.

1.5.3 Central Symmetry Isn't Everything: The Octagon Plus Two

Following this success, we might easily be tempted to conjecture that all centrally symmetric lattice sets have self-dual corresponding monomial varieties. Unfortunately, this isn't the case as the next example demonstrates.

By adding two points to the preceding example, we obtain another centrally symmetric lattice set

$$V = \{(0,0),(1,0),(3,0),(0,1),(2,1),(1,3),(3,3),(0,4),(2,4),(3,4)\},$$

pictured in Figure 1.6.

In fact, when we follow the same procedure as in the previous two examples, examining the matrices of k-Jets and their kernels as shown in Table 1.3, we find that $\dim(\ker J_k v)$ has generic value 1. However we see that two of the entries of $\ker J_k v$ are zero. If we look at the lattice points associated with $\ker J_k v$, we find

[1]This can be shown in two ways. First, if adding derivatives of order $k+1$ to $J_k v$ doesn't increase its rank, then the derivatives of order $k+1$ can be written as linear combinations of the derivatives of orders up to k. It's then clear that derivatives of, say, order $k+2$ depend on derivatives of order k. Second, we can show this result using Theorem 2.1.6 and the well-known fact that Hilbert functions of point sets are strictly increasing until they reach their maximal value.

Table 1.3: The Octagon Plus Two

$$v(x,y) = (1, x, x^3, y, x^2y, xy^3, x^3y^3, y^4, x^2y^4, x^3y^4)$$

k	$J_k(v)$	$\ker(J_k(v))$
0	$\begin{bmatrix} 1 & x & x^3 & y & x^2y & xy^3 & x^3y^3 & y^4 & x^2y^4 & x^3y^4 \end{bmatrix}$	$\begin{bmatrix} -x & -y & -x^3 & -x^2y & -xy^3 & -y^4 & -x^3y^3 & -x^2y^4 & -x^3y^4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
1	$\begin{bmatrix} 1 & x & x^3 & y & x^2y & xy^3 & x^3y^3 & y^4 & x^2y^4 & x^3y^4 \\ 0 & 1 & 3x^2 & 0 & 2xy & y^3 & 3x^2y^2 & 0 & 2xy^4 & 3x^2y^4 \\ 0 & 0 & 0 & 1 & x^2 & 3xy^2 & 3x^3y^2 & 4y^3 & 4x^2y^3 & 4x^3y^3 \end{bmatrix}$	$\begin{bmatrix} 2x^3 & 2x^2y & 3xy^3 & 3y^4 & 5x^3y^3 & 5x^2y^4 & 6x^3y^4 \\ -3x^2 & -2xy & -y^3 & 0 & -3x^2y^3 & -2xy^4 & -3x^2y^4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & -3xy^2 & -4y^3 & -3x^3y^2 & -4x^2y^3 & -4x^3y^3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
2	$\begin{bmatrix} 1 & x & x^3 & y & x^2y & xy^3 & x^3y^3 & y^4 & x^2y^4 & x^3y^4 \\ 0 & 1 & 3x^2 & 0 & 2xy & y^3 & 3x^2y^2 & 0 & 2xy^4 & 3x^2y^4 \\ 0 & 0 & 0 & 1 & x^2 & 3xy^2 & 3x^3y^2 & 4y^3 & 4x^2y^3 & 4x^3y^3 \\ 0 & 0 & 6x & 0 & 2y & 0 & 6xy^3 & 0 & 2y^4 & 6xy^4 \\ 0 & 0 & 0 & 0 & 2x & 3y^2 & 9x^2y^2 & 0 & 8xy^3 & 12x^2y^3 \\ 0 & 0 & 0 & 0 & 0 & 6xy & 6x^3y^3 & 12y^2 & 12x^2y^2 & 12x^3y^2 \end{bmatrix}$	$\begin{bmatrix} -4x^3y^3 & -3x^2y^4 & x^3y^4 & -6x^3y^4 \\ 4x^2y^3 & 2xy^4 & -x^2y^4 & 5x^2y^4 \\ 0 & 0 & -y^4 & 0 \\ 3x^3y^2 & 3x^2y^3 & -x^3y^3 & 5x^3y^3 \\ -3xy^2 & -y^3 & 3xy^3 & -3xy^3 \\ -x^2 & -2xy & -2x^2y & -2x^2y \\ 1 & 0 & 0 & 0 \\ 0 & 0 & x^3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
3	$\begin{bmatrix} 1 & x & x^3 & y & x^2y & xy^3 & x^3y^3 & y^4 & x^2y^4 & x^3y^4 \\ 0 & 1 & 3x^2 & 0 & 2xy & y^3 & 3x^2y^2 & 0 & 2xy^4 & 3x^2y^4 \\ 0 & 0 & 0 & 1 & x^2 & 3xy^2 & 3x^3y^2 & 4y^3 & 4x^2y^3 & 4x^3y^3 \\ 0 & 0 & 6x & 0 & 2y & 0 & 6xy^3 & 0 & 2y^4 & 6xy^4 \\ 0 & 0 & 0 & 0 & 2x & 3y^2 & 9x^2y^2 & 0 & 8xy^3 & 12x^2y^3 \\ 0 & 0 & 0 & 0 & 0 & 6xy & 6x^3y^3 & 12y^2 & 12x^2y^2 & 12x^3y^2 \\ 0 & 0 & 6 & 0 & 0 & 0 & 6y^3 & 0 & 0 & 6y^4 \\ 0 & 0 & 0 & 0 & 2 & 0 & 18xy^2 & 0 & 8y^3 & 24xy^3 \\ 0 & 0 & 0 & 0 & 0 & 6y & 18x^2y & 0 & 24xy^2 & 36x^2y^2 \\ 0 & 0 & 0 & 0 & 0 & 6x & 6x^3 & 24y & 24x^2y & 24x^3y \end{bmatrix}$	$\begin{bmatrix} x^3y^4 \\ -x^2y^4 \\ 0 \\ -x^3y^3 \\ xy^3 \\ x^2y \\ -y \\ 0 \\ -x \\ 1 \end{bmatrix}$

that only the 8 points making up the octagon are present in the lattice set of the dual variety, and we have "lost" the two points that we added.

The next question is under what conditions some of the points in a centrally symmetric variety are not present in its dual. The answer is somewhat surprising, and will be outlined in the next section, and fully demonstrated in Chapter 2.

1.6 A Sketch of the Results Using Examples

We will be able to give a complete characterization of self-dual affine monomial varieties in the next chapter. However, this characterization uses surprisingly complex tools, so we present the following analysis of the three previous examples in order to illustrate how we can determine self-duality for an arbitrarily chosen variety. The theorem we will eventually use is as follows:

A given affine monomial variety v is self-dual if its lattice set V satisfies the following conditions:

1. V is centrally symmetric.

2. Consider the codimension of the linear space of algebraic curves vanishing on the lattice set V. By this we mean the total number of linearly independent polynomials of some fixed degree k or less[2] minus the number of such polynomials vanishing on V. In order for V to correspond to a self-dual variety, there must be a k such that this codimension is equal to the number of points in the lattice set minus one. The sequence of codimensions for all k is known as the Hilbert function of the lattice set, denoted H_V. Thus our condition is: $\exists k : H_V(k) = |V| - 1$.

   Examples:

   (a) As we can see, the Octagon shown in Figure 1.5 lies on three linearly independent cubics (actually the product of the unique conic passing through the eight points of the Octagon with any three linearly independent lines in the plane). Since there are 10 total linearly independent cubics, and three linearly independent ones vanishing on the octagon we see that the codimension of cubics vanishing on V is $10 - 3 = 7$. Since there are eight points in the set and the codimension of cubics that are simultaneously zero on v is 7, Condition 2 is satisfied. We can see an abbreviated form of this information in Table 1.4.

   (b) Similarly, since there are ten points in the Octagon Plus Two pictured in Figure 1.6, and there are ten total linearly independent plane cubics, and additionally we find that there is a unique cubic through all ten points (shown in Figure 1.7), the codimension of cubics vanishing on V is $10 - 1 = 9$, and so Condition 2 is again satisfied. This information is presented in Table 1.5.

[2]The number of such linearly independent polynomials of degree $\leq k$ in $\mathbb{C}^n$ is given by $\binom{n+k}{k}$.

Table 1.4: Curves through the Octagon

		lines	conics	cubics	quartics	$\cdots$	deg k
total in $\mathbb{R}^2$	1	3	6	10	15		$\binom{k+2}{2}$
0 on V	0	0	1	3	7		$\binom{k+2}{2} - 8$
$\dim(J_k) = H_V$	1	3	5	7	8		8
$\dim(\ker J_k)$	7	5	3	1	0		0

Table 1.5: Curves through the Octagon Plus Two

		lines	conics	cubics	quartics	$\cdots$	deg k
total in $\mathbb{R}^2$	1	3	6	10	15		$\binom{k+2}{2}$
0 on V	0	0	0	1	5		$\binom{k+2}{2} - 10$
$\dim(J_k) = H_V$	1	3	6	9	10		10
$\dim(\ker J_k)$	9	7	4	1	0		0

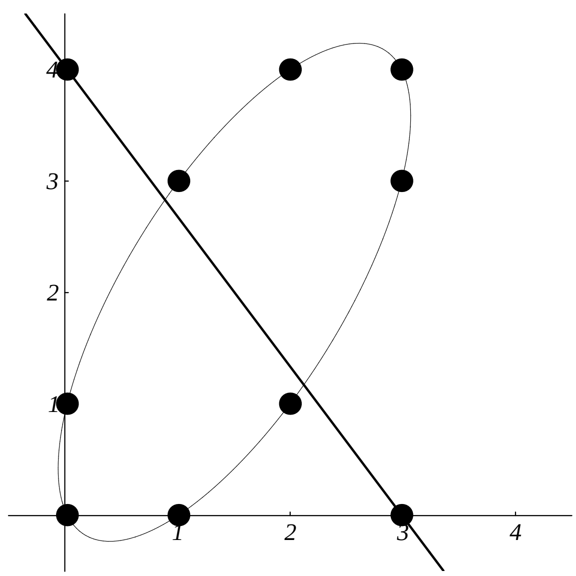


Figure 1.7: The Octagon Plus Two Points and its Unique Cubic

Table 1.6: Hilbert functions: The Octagon Plus Two

k	0	1	2	3	4	$\cdots$	$k > 4$
$H_V(k)$	1	3	6	9	10		10
$H_{V\setminus\{m\}}(k)$	1	3	6	8	9		9

3. For every point $m \in V$, there are no more curves of degree $\leq k$ vanishing on $V \setminus \{m\}$ than vanish on V. Put more succintly, $H_{V\setminus\{m\}}(k) = H_V(k)$ for all $m \in V$.

   Examples:

   (a) As we can see in the case of the Octagon, removing a point doesn't allow any additional cubics to pass through the remaining seven points– there is enough redundancy in the points that any $(|V|-1)$-size subset determines precisely the same curves of degree 3 as the full set. This condition will be clearer when we examine a variety that does not satisfy it, such as the one below.

   (b) In the case of the Octagon Plus Two, removing one of the two points not in the Octagon results in too many cubics passing through the remaining points. What is happening is that the removal of one of the two "outlying" points allows the remaining points to be covered by the product of the conic through the points in the Octagon and any line through the additional point. Since there are two linearly independent lines through the remaining "outlying" point, we see that there are too many cubics passing through the reduced set, and thus the variety will not be self-dual. If we examine the Hilbert function of the lattice set, and of the lattice set minus one of the "outliers" m, there's not enough redundancy in V. In Table 1.6, we see that $H_{V\setminus\{m\}}(3) < H_V(3)$[3]. Note, however, that even though this variety is not self-dual, its dual is the Octogon, which is self-dual as we have just seen– in fact, the dual of any variety is itself dual.

We have thus succeeded in restating our problem: We are looking for all centrally symmetric subsets V of $\mathbb{Z}^n_{\geq 0}$ which include the origin and for which $H_V(k) = |V| - 1$ and $H_{V\setminus\{m\}}(k) = |V| - 1$ for some k and all $m \in V$.

In the next chapter we present proofs of the claims we have just made and a theorem which is useful in the search for self-dual varieties. The final chapter of the thesis presents all the examples we have found so far.

---

[3]Note here that m can be either of the two "outlying" points, $(3, 0)$ or $(0, 4)$.

Chapter 2

The Results

As shown in the last chapter, self-dual affine monomial varieties can be characterized in terms of central symmetry and certain conditions on the Hilbert functions of their exponent sets. The results that make up this description are stated and proved in this chapter. Also, we show that given any affine monomial variety that has a dual, the dual is itself self-dual. In addition, a result is presented that tells us that the "generic" centrally symmetric lattice set additionally satisfies one of the other necessary criteria for self-duality.

2.1 A Characterization of the Self-Duals

The results of this chapter concern a symmetry condition on the lattice sets of self-dual monomial variety, and a condition on their Hilbert functions. We begin with the results on symmetry.

2.1.1 Theorems about Central Symmetry

Our first result shows that we can precisely determine all the monomials that potentially can be in the kernel of a given variety's matrix of k-jets, assuming that it does have a dual.

Theorem 2.1.1. *Let $v(x) = (1, x^{m_1}, \ldots, x^{m_t})$ be a standard affine monomial variety. Suppose there exists an integer k such that the dimension of the kernel of the matrix of k-jets, $J_k v$, is generically 1. Then, generically,*

$$\ker(J_k v(x)) = \mathrm{Span}\{a_0 x^{m_t}, a_1 x^{m_t - m_1}, \ldots, a_{t-1} x^{m_t - m_{t-1}}\}$$

for some constants $a_0, \ldots, a_{t-1}$.

Proof. Since the dimension of $\ker(J_k v)$ is 1, we see that there are t linearly independent rows in $J_k v$, as there are $t+1$ columns. For each a_j, multiply the row corresponding to the a_j-th derivative by x^{a_j}, so that the i-th column only contains multiples of $x^{m_{i+1}}$. Hence, performing row operations and discounting rows of zeros yields:

$$
\begin{aligned}
J_k v(x) &\sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_0 x^{m_t} \\ 0 & x^{m_1} & 0 & \cdots & 0 & a_1 x^{m_t} \\ 0 & 0 & x^{m_2} & \cdots & 0 & a_2 x^{m_t} \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & x^{m_{t-1}} & a_t x^{m_t} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_0 x^{m_t} \\ 0 & 1 & 0 & \cdots & 0 & a_1 x^{m_t - m_1} \\ 0 & 0 & 1 & \cdots & 0 & a_2 x^{m_t - m_2} \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_t x^{m_t - m_{t-1}} \end{bmatrix}.
\end{aligned}
$$

From this we can directly read off the kernel, which will be in the form:

$$\ker(J_k v(x)) = \mathrm{Span}\{(-a_0 x^{m_t}, -a_1 x^{m_t - m_1}, -a_2 x^{m_t - m_2}, \ldots, -a_t x^{m_t - m_{t-1}}, 1)\}.$$

□

Example: For instance, let $v(x, y) = (1, x, xy, y^2)$. The matrix of 1-jets reduces to:

$$J_1 v = \begin{bmatrix} 1 & x & xy & y^2 \\ 0 & 1 & y & 0 \\ 0 & 0 & x & 2y \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & y^2 \\ 0 & 1 & 0 & -2\frac{y^2}{x} \\ 0 & 0 & 0 & 2\frac{y}{x} \end{bmatrix}.$$

As long as $x \neq 0$, the kernel of $J_1 v$ has dimension 1, and can be parametrized by

$$(x, y) \mapsto (-y^2, 2\frac{y^2}{x}, 2\frac{y}{x}, 1).$$

Since the kernel is a linear space, we can clear denominators to express the kernel of $J_1 v$ as an affine monomial variety (at those points where $x \neq 0$):

$$(x, y) \mapsto (-xy^2, 2y^2, 2y, x).$$

Of course, this variety is not self-dual, but this parametrization of its dual suggests a more precise definition of the dual variety when it exists.

Definition 2.1.2. *With the notation and hypotheses of the preceding theorem, the* **dual** *of v is defined to be the affine monomial variety*

$$
\begin{aligned}
\mathrm{Dual}_k v(x) &= x^l (a_0 x^{m_t}, a_1 x^{m_t - m_i}, \ldots, a_{t-1} x^{m_t - m_{t-1}}, 1) \\
&= (a_0 x^{m_t + l}, a_1 x^{m_t - m_1 + l}, \ldots, a_{t-1} x^{m_t - m_{t-1} + l}, x^l)
\end{aligned}
$$

where $l \in \mathbb{Z}^n_{\geq 0}$ is chosen with each coordinate as small as possible subject to the condition that $m_t - m_1 + l, \ldots, m_t - m_{t-1} + l$ are in $\mathbb{Z}^n_{\geq 0}$.

The fact that this definition of the dual is consistent with the original definition given in Chapter 1 in terms of hyperplanes containing osculating spaces is clear in light of Theorem 2.1.1. For a given affine monomial variety v, when there exists a k such that $\dim(\ker(J_k v)) = 1$, the dual as given above will be an affine monomial variety parametrizing the hyperplane containing $\mathrm{Osc}_k v(x)$ at all points x (except for possibly some set of measure zero).

We now show that self-dual affine monomial varieties have centrally symmetric corresponding lattice sets as a corollary to Theorem 2.1.1. First we formally define central symmetry.

Definition 2.1.3. *A lattice set V is* **centrally symmetric** *if $m \in V \Leftrightarrow 2c - m \in V$ for all $m \in V$ where the midpoint of the lattice set is given by $c := \frac{1}{|V|}\sum_{m\in V} m$. An equivalent condition is that the sum of any point and its opposite be twice the midpoint.*

Corollary 2.1.4. *If an affine monomial variety v is self-dual, then its corresponding lattice set V is centrally symmetric.*

Proof. We choose to put v in standard form, arranging its monomials in ascending lexicographic order and discarding zero components. Then, since an affine monomial variety is self-dual exactly when its dual exists and they are the same up to a toric change of coordinates, we can see that the self-duality condition can be expressed as:

$$(1, x^{m_1}, \ldots, x^{m_t}) = x^l(a_0 x_t^m, a_1 x^{m_t - m_1}, \ldots, a_{t-1}x^{m_t - m_{t-1}}, 1).$$

This can only happen when $l = 0$, since otherwise there will be no constant component in the dual to correspond to the 1 component in the original variety. Note now that none of the a_i can be zero, since the variety is self-dual. We apply a toric change of coordinates and set $a_i = 1$ for all i. Since the original m_i were arranged in increasing order, we can see that the components of the dual are listed in decreasing order. Thus the i-th component of v will correspond to the $(t-i)$-th component of $\mathrm{Dual}_k v$, and we obtain the condition that $x^{m_t - m_i} = x^{m_{t-i}}$. This, in turn, directly implies that

$$m_i + m_{t-i} = m_t,$$

which is precisely the condition that the corresponding lattice set V be centrally symmetric with center $m_t/2$. □

Remark: We offer a geometric interpretation of what this corollary implies for self-dual (and certain non-self-dual) varieties. If there is a dual for a given variety (that is, if the nullity of $J_k v$ reaches 1 for some k), then the exponents of the duals will be the "flips" of the original exponents through the "center of mass" $m_t/2$. Of course, some of the a_i may be 0, so certain pairs of exponents may disappear in taking the dual. We will see explicit conditions for these two possibilities in the theorems just ahead. We thus turn to:

2.1.2 Theorems about Hilbert Functions

The crux of the remaining results is the relationship between the ranks of the matrices of partial derivatives and the Hilbert function of the lattice set. For those who are not familiar with Hilbert functions, any textbook on algebraic geometry will be useful, such as [1] or [4]. The definition is given below.

Definition 2.1.5. *The* **Hilbert function** *of a set of lattice points $V \subset \mathbb{Z}^n \subset \mathbb{C}^n$ is defined to be the codimension in the linear space of polynomials of degree less than or equal to k of those polynomials satisfied by V.*

Theorem 2.1.6 ([6])**.** *The generic rank of the matrix of k-jets of an affine monomial variety v is given by the value of the Hilbert function of its lattice set V at k,*

$$\operatorname{rank}(J_k v) = H_V(k).$$

Proof. By examining the formula for the partial derivative of a monomial x^ℓ:

$$\begin{aligned} \frac{1}{a!} x_a^\ell &= \frac{1}{a!} \frac{\partial^{|a|} x^\ell}{\partial x^{a_1} \cdots \partial x^{a_n}} \\ &= \binom{\ell_1 \cdots \ell_n}{a_1 \cdots a_n} x^{\ell - a} \\ &= \binom{\ell}{a} x^{\ell - a}, \end{aligned}$$

we see that the column of $J_k v$ corresponding to the monomial x^{m_i} has the form

$$\binom{m_i}{a}_{0 \le |a| \le k} = \left[\binom{m_{i,1}}{a_1} \cdots \binom{m_{i,n}}{a_n} \right]_{0 \le |a| \le k},$$

after multiplying the a-th row of $J_k v$ by $\frac{1}{a!}$. But $\left\{ \binom{x_1}{a_1} \cdots \binom{x_n}{a_n} \right\}_{0 \le |a| \le k}$ forms a basis for the space of polynomials of degree $\le k$ in $x_1, \ldots, x_n$. Thus, the linear relations on the rows of $J_k v$ correspond to polynomials of degree $\le k$ passing through V. Finally, we note that v attains its generic rank at the point $(1, \ldots, 1)$, since the minor determinants of $J_k v$ are monomials. □

Finally we offer a complete characterization of self-dual affine monomial varieties.

Theorem 2.1.7. *An affine monomial variety v is self-dual if and only if its lattice set V satisfies the following properties:*

1. *The lattice set V is centrally symmetric.*
2. *There is a $k \in \mathbb{N}$ such that $H_V(k) = |V| - 1$.*
3. *There is no $m \in V$ such that $H_{V \setminus \{m\}}(k) < |V| - 1$.*

Proof. The necessity of the first property is simply Theorem 2.1.4.

The necessity of the second property follows from the fact that for the dimension of $\mathrm{Dual}_k v$ to be 1 for some k, we must have $H_V(k) = |V| - 1$ by the preceeding result.

The necessity of the third property is less obvious. Proceeding indirectly, we assume that $\dim(\mathrm{Dual}_k v) = 1$ for some k, and that one of the components of $\mathrm{Dual}_k v$ is zero. But the components of $\mathrm{Dual}_k v$ are the determinants of the minors consisting of t linearly independent rows of $J_k v$, modulo Lemma 2.1.8. If a component is zero, then some minor determinant is zero, thus the rank of one of these minor determinants is less than the rank of the full matrix of k-jets. Therefore the Hilbert function of V minus the point left out corresponding to the deletion of the i-th column (i.e. $H_{V\setminus\{m_i\}}$) has dropped below the value of the Hilbert function of V. This establishes necessity.

Conversely, if the lattice set is centrally symmetric and the nullity of $J_k v$ reaches 1 for some k, then we can find a dual using Theorem 2.1.1, and furthermore the lattice points will all be in $\ker J_k v$ if their coefficients are not zero. When the Hilbert function of $V \setminus \{m_i\}$ is the same as the Hilbert function of V for all m_i, none of the minor determinants can be zero, and thus none of the coefficients can be zero by the reverse of the argument of the necessity of Condition 3. This completes the proof. □

There remains only to prove the following technical result:

Lemma 2.1.8. *Given an $m \times n$ matrix A with rank $n-1$, let $\tilde{A}$ be an $(n-1)\times n$ matrix consisting of any $n-1$ linearly independent rows of A. Then*

$$\ker(A) = \mathrm{Span}\{(c_1, \ldots, c_n)\},$$

where c_j is the determinant of the $(n-1)\times(n-1)$ submatrix of $\tilde{A}$ obtained by deleting the j-th column of $\tilde{A}$.

Proof. For a fixed $i \in [1, n-1]$, let D_i be the $n \times n$ matrix constructed by appending the i-th row of $\tilde{A}$ to the bottom of $\tilde{A}$. Then clearly $\det(D_i) = 0$ since it contains the same row twice. Performing cofactor expansion along this additional bottom row, we see that

$$0 = \det(D_i) = \sum_{k=1}^{n} (-1)^{n+k} \tilde{a}_{ik} c_k.$$

However, this equation can also be interpreted as the explicit formulation of a matrix multiplication: the product of the vector of determinant minors and any fixed row of $\tilde{A}$ is zero. Thus $(c_1, \ldots, c_n) \subset \ker(A)$, and since we know that $\mathrm{rank}(A) = n-1$, the result follows. □

2.2 Duals are Self-Dual

In Sections 1.5.2 and 1.5.3 we saw that when comparing an affine monomial variety to its dual, we sometimes lose elements of the lattice set. The remaining lattice points generate a self-dual affine monomial variety. In other words, if a variety has a dual, then that dual is self-dual, as the following result shows.

Theorem 2.2.1. *Suppose V is centrally symmetric, $H_V(k) = |V| - 1$, and let $W = \{m \in V : H_{V\setminus\{m\}}(k) < |V| - 1\}$. Then $V \setminus W = \mathrm{Dual}_k v$ generates a self-dual affine monomial variety.*

Proof. First, note that $V \setminus W$ is centrally symmetric since H_V is invariant under the change of coordinates taking each point to its reflection through the midpoint of V.

Now, let v and w be the standard affine monomial varieties corresponding to V and $V \setminus W$ respectively. Then, by the proof of Theorem 2.1.7, $\mathrm{Dual}_k v$ will have zero entries in the components corresponding to the elements of W. Let $L = \mathrm{Dual}_k v$ with its zero entries removed. Then L, considered as a column vector, lies in $\ker J_k w$, since its product with each row of $J_k w$ is zero.

On the other hand, if there were another element in a basis for $\ker J_k w$ then this vector with zeros added in entries corresponding to elements of W would lie in $\ker J_k v$ and would be linearly independent from $\mathrm{Dual}_k v$, which would be a contradiction. We thus see that $\dim(\ker J_k w) = 1$, and since an element of $\ker J_k w$ has no zero entries by construction, the proof of Theorem 2.1.7 shows that w is a self-dual affine monomial variety. □

2.3 A Theorem For Finding Self-Dual Varieties

According to the next theorem, the "generic" affine monomial variety will have the correct Hilbert function– it will reach the number of points in the lattice set minus 1. This does not guarantee that Condition 3 of Theorem 2.1.7 will be satisfied, but the construction used in the proof allows for a constructive search for self-duals.

Theorem 2.3.1 (Reichstein). [1]

1. *Fix an odd integer $k = 2d+1$. Then for a generic choice of r points $p_1, \dots, p_r$ in $\mathbb{C}^2$ we have $H_V(k-1) = |V| - 1 = 2r - 1$ where $r = d^2 + d + 1$ and $V = \{p_1, \dots, p_r, -p_1, \dots, -p_r\}$.*
2. *Similarly, if $k = 2d$ is an even integer, for a generic choice of r points $p_1, \dots, p_r$ in $\mathbb{C}^2$ we have $H_V(k-1) = |V| - 1 = 2r - 1$ where $r = d^2 + 1$ and $V = \{p_1, \dots, p_r, -p_1, \dots, -p_r\}$.*

Proof. Only the case where k is odd, $k = 2d + 1$, will be treated, as the case for even k is similar.

First we make the definitions:

$$\begin{aligned} W &:= \mathrm{Span}\{x^m y^n \subset \mathbb{C}[x,y] : m + n \le k - 1 = 2d\} \\ W_0 &:= \mathrm{Span}\{x^m y^n \subset W : m + n \text{ is odd}\} \\ W_1 &:= \mathrm{Span}\{x^m y^n \subset W : m + n \text{ is even}\}. \end{aligned}$$

[1] This result was received in private communication from Zinovy Reichstein at OSU.

Since for each i there are exactly $i+1$ monomials of degree i, we can see that

$$\begin{aligned}
\dim W_0 &= 1+3+\cdots+2d+1 = (d+1)^2 \\
\dim W_1 &= 2+4+\cdots+2d = d(d+1) \\
\dim W &= 1+2+\cdots+2d+1 = (d+1)(2d+1) = \dim W_0 + \dim W_1.
\end{aligned}$$

We make the notational convention that for any set of points S, $W(S)$ will be the subspace of those elements in W vanishing on S, and similarly for $W_0(S)$ and $W_1(S)$.

We make the following two observations:

1. If S is symmetric about the origin then $W(S)$ is a direct sum of $W_0(S)$ and $W_1(S)$. In particular,
$$\dim W(S) = \dim W_0(S) + \dim W_1(S).$$
2. For an arbitrary subset $S \subset \mathbb{C}^2$,
$$W_0(S) = W_0(S \cup -S) \text{ and } W_1(S) = W_1(S \cup -S).$$

Now we are ready to construct V. Let $i = d(d+1) = \dim W_1 < \dim W_0$. For a generic choice of distinct points $p_1, \ldots, p_i \subset \mathbb{Z}^2$, we have:

$$\begin{aligned}
\dim W_0(p_1,\ldots,p_i) &= \dim W_0 - i &&= (d+1)^2 - i \\
\dim W_1(p_1,\ldots,p_i) &= \dim W_1 - i &&= d(d+1) - i = 0
\end{aligned}$$

Combining the above two observations we see that

$$\begin{aligned}
\dim W(p_1,\ldots,p_i,-p_1,\ldots,-p_i) &= \dim W_0(p_1,\ldots,p_i) + \dim W_1(p_1,\ldots,p_i) \\
&= (\dim W_0 - i) + (\dim W_1 - i) \\
&= \dim W - 2i
\end{aligned}$$

Now we add one more point p_r where $r = \dim W_1 + 1 = d(d+1)+1 = d^2+d+1$. If we choose it generically, it will reduce the dimension of W_0 by 1. On the other hand, it cannot reduce the dimension of W_1 any more because $W_1(p_1,\ldots,p_i)$ is already 0-dimensional. Thus the two points p_r and $-p_r$ impose only one additional linear condition on W which causes the Hilbert function to be 1 less than its maximal value. More precisely, if $V = (p_1,\ldots,p_r,-p_1,\ldots,-p_r)$, then

$$\begin{aligned}
\dim W(V) &= \dim W_0(p_1,\ldots,p_r) + \dim W_1(p_1,\ldots,p_r) \\
&= (\dim W_0 - r) + 0 = \dim W_0 - r,
\end{aligned}$$

and thus

$$\begin{aligned}
H_V(k-1) &= \dim W - \dim W(V) = \dim W - \dim W_0 + r \\
&= \dim W_1 + r = 2r - 1 = |V| - 1.
\end{aligned}$$

N.B. It will be important to note that we can define what we mean by "generic" rather specifically: we construct the set of $\{p_1,\ldots,p_r\}$ by choosing p_i so that it is not in $\{0, p_1, \ldots, p_{i-1}\}$ and does not lie in the zero loci of two polynomials which we can choose at random in $W_0(p_1,\ldots,p_{i-1})$ or $W_1(p_1,\ldots,p_{i-1})$. $\square$

Chapter 3

Examples of Self-Dual Affine Monomial Varieties

This chapter consists of a list all of the self-dual varieties that we have found to date. We are primarily interested in the standard, smooth, smoothly embedded affine monomial varieties, since these are the ones that when glued together will form well-behaved toric varieties. In order to see why this is true, we refer the reader to a text on toric varieties such as [2] or [5].

3.1 The Self-Duals So Far

Definition 3.1.1. *An affine monomial variety v is* **smoothly embedded** *if each side of the convex hull of its associated lattice set V contains the first lattice point on that side from each of the vertices.*

We see in Figure 3.1 an example of a lattice set generating a smoothly embedded variety, and one that generates a variety that is not smoothly embedded. In particular, for the non-smoothly embedded lattice set, notice that the points $(1, 2)$ and $(3, 2)$ are missing along the long sides of the convex hull.

Definition 3.1.2. *An affine monomial variety v is* **smooth** *if for each vertex of the convex hull of V the first lattice points along the adjacent edges form a basis for the lattice when translated to the origin.*

In Figure 3.2 we see two lattice sets which respectively generate smooth and non-smooth affine monomial varieties. In particular for the non-smooth lattice, notice that the set of vectors $\{(1, 0), (1, 2)\}$ corresponding to the first points along the edges from the origin do not form a basis for $\mathbb{Z}^2$.

3.2 Examples of Self-Dual Varieties

We list the self-duals we have found so far:

Figure 3.1: A smoothly embedded lattice set and one that isn't

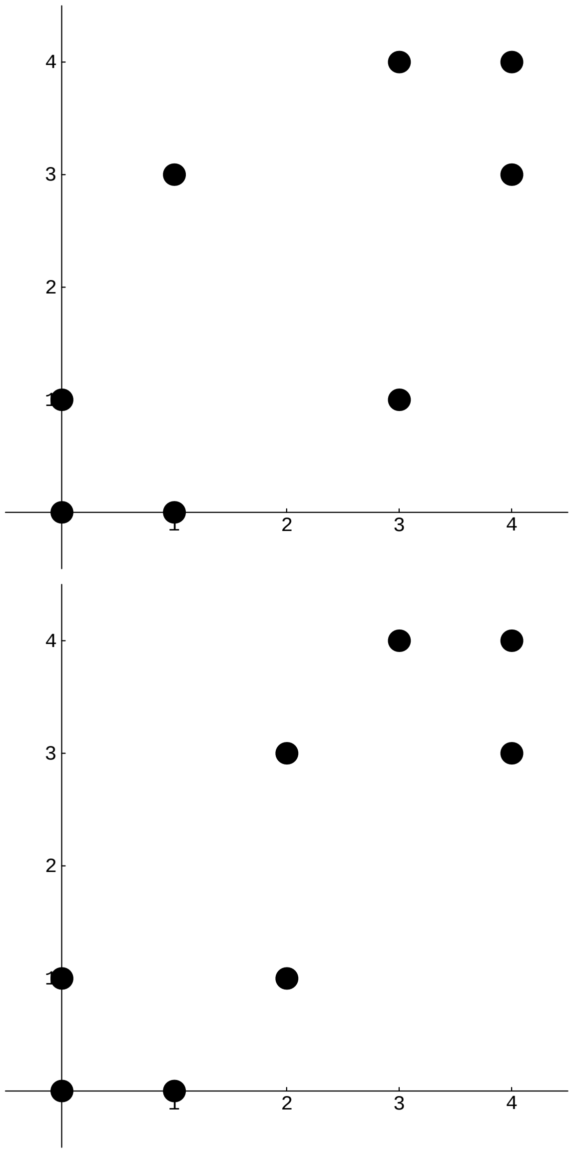
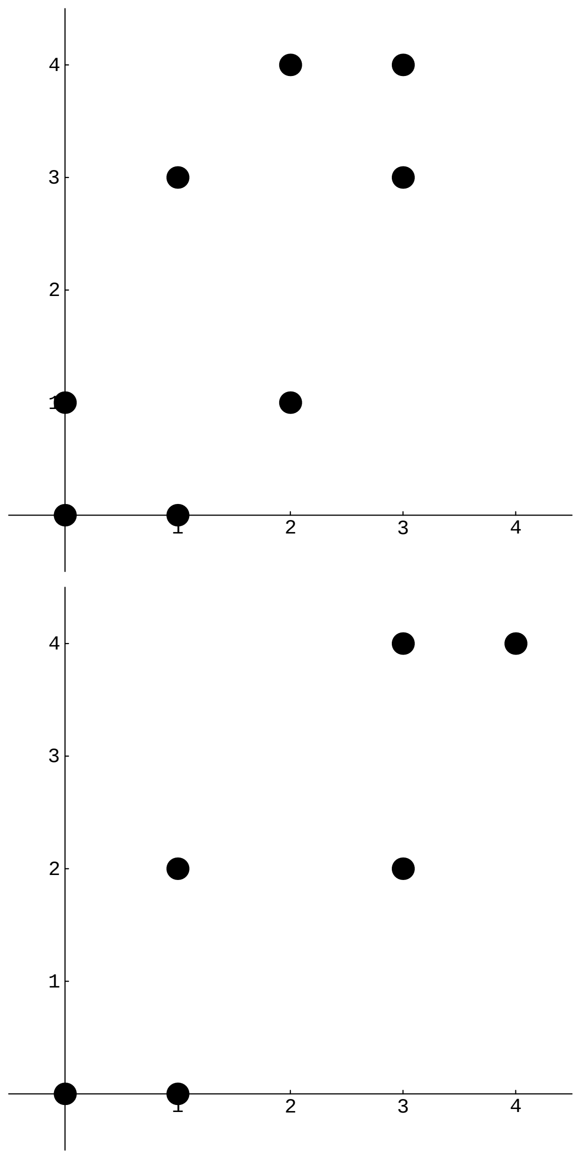


Figure 3.2: A smooth lattice set and one that isn't

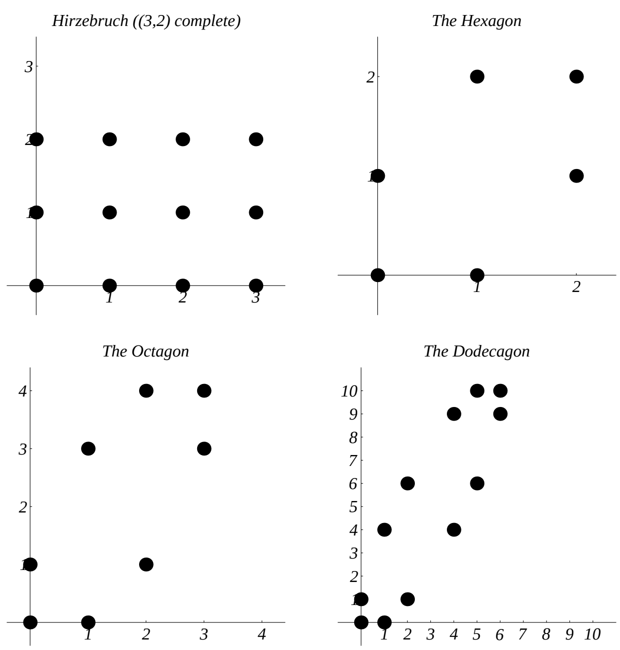


1. Any curve $v : \mathbb{C} \to \mathbb{C}^{t+1}$ whose lattice set is centrally symmetric. This is easy to see because n collinear points have no gaps in their Hilbert functions.

2. The three surfaces plus one infinite class of surfaces that were known before the thesis, shown in Figure 3.3. They were found in [6] in conjunction with studies of the inflectionary behavior of affine monomial varieties.

   - Rectangles and parallelograms which include all of their interior points (these are very special cases of *Hirzebruch surfaces*, in fact called *Segre embeddings* of $\mathbb{P}^1 \times \mathbb{P}^1$).
   - $V = \{(0,0),(1,0),(2,1),(2,2),(1,2),(0,1)\}$
   - $V = \{(0,0),(1,0),(2,1),(3,3),(3,4),(2,4),(1,3),(0,1)\}$
   - $V = \begin{array}{c}\{(0,0),(1,0),(2,1),(4,4),(5,6),(6,9),(6,10),\\ (5,10),(4,9),(2,6),(1,4),(0,1)\}\end{array}$

Figure 3.3: The four original surfaces



3. Theorem 2.3.1 implies that the generic centrally symmetric lattice set will satisfy Condition 2 of Theorem 2.1.7, provided it contains the right number of points. Thus we need only construct a class of lattice sets that generically satisfies Condition 3.

   In fact, we have found an infinite class of smooth, smoothly embedded, self-dual affine monomial varieties. For any $(b, c, d, e) \in \mathbb{Z}^4$ with $c > b > 1$ and

$d > e > 1$, let

$$V = \begin{array}{r} \{(0,0),(1,0),(0,1),(b,1),(c-1,e),(c,d-1), \\ (c,d),(c-1,d),(c-b,d-1),(1,d-e)\}. \end{array}$$

By construction, the variety corresponding to this lattice set is smooth for all values of (b,c,d,e). It is also smooth by construction at all vertices except $(c-1,e)$ and its opposite, $(1,d-e)$. The condition that it be smooth at these two points is given by:

$$\begin{vmatrix} 1 & d-1-e \\ b-c+1 & 1-e \end{vmatrix} = 1 \Rightarrow e = \frac{(b-c+1)(d-1)}{b-c}.$$

We will show that the one-parameter set generated by

$$(b,c,d,e) = (2,5,1+3k,2k)$$

is self dual for each value of k. By substitution it satisfies the above smoothness condition for all $k > 0$.

To check self-duality, we verify Conditions 1,2, and 3 of Theorem 2.1.7. By construction, V is centrally symmetric, so Condition 1 is satisfied. To see condition 2, use Maple or another computer algebra system to check that there is a unique irreducible cubic through V. (To show irreducibility we checked that the cubic's projective closure is actually smooth.) The uniqueness of the cubic implies that

$$H_V(3) = 10 - 1 = 9 = |V| - 1.$$

Thus Condition 2 is satisfied. Finally, by the following theorem, Condition 3 is satisfied generically.

Theorem 3.2.1. *If there is a unique, irreducible cubic C through the centrally symmetric points $V = \{p_1, \ldots, p_{10}\}$, then C is the unique cubic passing through $V \setminus \{p_i\}$ for $1 \le i \le 10$.*

Proof. First we translate V so it is centered at the origin, since this will not affect the number of cubics passing through the points.

Let $\tilde{C}$ be a cubic vanishing on $V \setminus \{p_i\}$. First we show that $\tilde{C}$ is odd.

Write $\tilde{C} = \tilde{C}_O + \tilde{C}_E$ as a sum of odd and even components. Then both $\tilde{C}_O$ and $\tilde{C}_E$ vanish on the set $V \setminus \{p_i, -p_i\}$. Since $\tilde{C}_E$ meets C at the eight points $V \setminus \{p_i, -p_i\}$, and $\deg(\tilde{C}_E) \le 2$, Bézout's Theorem implies that if $\tilde{C}_E \ne 0$ then it is a factor of C. By assumption C is irreducible, hence $\tilde{C}_E = 0$ and $\tilde{C}$ is odd.

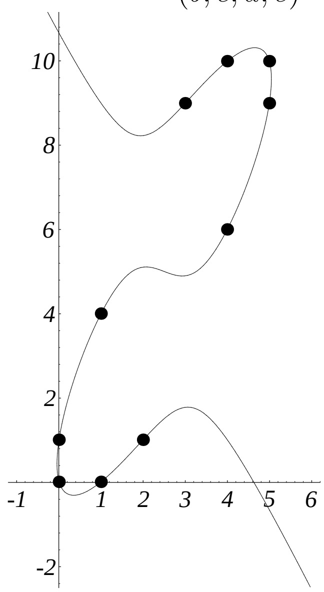
Now we will show that $\tilde{C}$ passes through V. $\tilde{C}$ is satisfied by $V \setminus \{p_i\}$, hence it vanishes on $-p_i$. But since $\tilde{C}$ is odd, $\tilde{C}(p_i) = -\tilde{C}(-p_i) = 0$. Thus C meets $\tilde{C}$ at 10 points, and two cubics can meet at no more than 9 points or they share a common component, again by Bézout's Theorem (see [1]). Therefore, since C is irreducible, $C = \tilde{C}$. $\square$

The following is an example of a 10-point set constructed as above: for the choice of parameters $(b,c,d,e) = (2,5,10,6)$, we find that the resulting lattice set is

$$V = \{(0,0),(1,0),(0,1),(2,1),(4,6),(5,9),(5,10),(4,10),(3,9),(1,4)\}.$$

This lattice set is pictured in Figure 3.4, along with the unique cubic vanishing on it.

Figure 3.4: V and the unique cubic vanishing on V
$(b,c,d,e) = (2,5,10,6)$



Using the ideas outlined above, one should be able to show that the following 2-parameter set of 10 lattice point sets gives rise to smooth, smoothly embedded, self-dual varieties:

$$(b,c,d,e) = (2,j,1+(j-2)k,(j-3)k),$$

though we have not checked the details thoroughly yet.

References

[1] William Fulton, *Algebraic curves*, Addison-Wesley Inc., 1989.

[2] ———, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, 1993.

[3] Oliver Gugenheim, *Inflections of parameterized surfaces*, Reed College Thesis, 1993.

[4] Joe Harris, *Algebraic geometry: A first course*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, 1992.

[5] Tadao Oda, *Convex bodies and algebraic geometry: an introduction to the theory of toric varieties*, A series of modern surveys in mathematics, vol. 15, Springer-Verlag, 1987.

[6] David Perkinson, *Inflections of toric varieties*, University of Oslo Preprint Series, 1994.