Wreath-Product Polytopes

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Table of Contents

List of Tables

List of Figures

Abstract

Given a finite group G and a real matrix representation $\rho(G) \subset GL(m, \mathbb{R})$, we define the polytope $P(G, \rho)$ to be the convex hull of the elements of $\rho(G)$, seen as points in $\mathbb{R}^{m^2}$. If $H \subseteq S_m$ is a permutation group, let $G \wr H$ be the wreath product of G and H. A representation of G and the natural permutation representation of H combine to give us a representation ρ of $G \wr H$. This thesis describes the combinatorial structure of the polytope $P(G \wr H, \rho)$ in the case that H is a regular permutation group.

Introduction

Given a finite group G and a real matrix representation $\rho\colon G \to GL(m, \mathbb{R})$, we can view $\rho(G)$ as a set of points in $\mathbb{R}^{m^2}$. This observation forms the basis for this thesis by allowing us to view a group as a geometric object. We construct the polytope $P(G, \rho) \subset \mathbb{R}^{m^2}$ by taking the convex hull of the set $\rho(G)$. The main goal of the study of such group polytopes is to relate the structure of the group G to the structure of its associated polytope $P(G, \rho)$.

This geometric construction is usually applied to a permutation group $G \subseteq S_m$ and the standard permutation representation $\rho\colon S_m \to GL(m, \mathbb{R})$. Ideally, we would be able to make observations about the general permutation polytope $P(G, \rho)$, but this proves to be surprisingly difficult [$\mathcal{M}$]. Instead, most of the literature makes an effort to characterize the combinatorial structure of $P(G, \rho)$ for a particular type of permutation group, G. Birkhoff [Bi], for example, characterizes the polytope for the full symmetric group, $P(S_m, \rho)$ as exactly the set of bistochastic $m \times m$ matrices. Brualdi [Br] and Gibson [BrG] have written several papers on the structure of this polytope. Interesting work has also been done by Onn [$\mathcal{O}$] and Billera and Sarangarajan [BiS]. There remain, however, several open questions about the so-called Birkhoff polytope, such as the derivation of the volume of $P(S_n, \rho)$ for arbitrary n [BP]. As another example, Perkinson and Collins [PC] characterize the Frobenius polytope $P(F, \rho)$, where F is a Frobenius group. This thesis works in the same vein by characterizing the polytope $P(G \wr H, \rho)$ of the wreath product $G \wr H$.

The first chapter outlines some basic results about general polytopes. Section 1.1 consists mainly of preliminary definitions and constructions, most of which can be found in [$\mathcal{Z}$]. Here we formally define convex hull, polytope, face lattice, and polar dual. In section 1.2 we examine three standard binary operations on polytopes: the cartesian product ($\times$), the direct sum ($\oplus$), and the free join ($\bowtie$). We prove the useful result $P^* \times Q^* \cong (P \oplus Q)^*$, where P and Q are polytopes containing the origin in their relative interiors, and '$*$' represents the polar dual operation. In addition, we investigate the combinatorial structure of $P \times Q$, $P \oplus Q$, and $P \bowtie Q$.

Chapter two discusses polytopes arising from groups. The first section consists of a review of real group representations and permutation representations, and how these are turned into polytopes. In section 2.2, some important results about general group polytopes are presented. Of fundamental importance is theorem 17, which states that the vertices of a polytope $P(G, \rho)$ are precisely the set $\rho(G)$. Section 2.3, then, talks about the computation of specific polytopes using computers, and gives some data on permutation group polytopes up to degree seven.

In the third chapter we finally address the wreath-product polytope $P(G \wr H, \rho)$. First (section 3.1) the wreath product of groups is defined, and some general structural attributes of the operation are discussed. Section 3.2 presents our main result that, given a representation of a group G, a regular permutation group $H \subseteq S_n$, and a representation $\rho(G \wr H)$, the wreath-product polytope $P(G \wr H, \rho)$ is isomorphic to $[P(G)^n]^{\oplus |H|}$ if the origin is in the relative interior of $P(G)$, or $[P(G)^n]^{\bowtie |H|}$ if not.

Chapter 1

Polytopes

1.1 Elementary Results

This section summarizes the basic theory of polytopes. A more detailed treatment can be found in [$\mathcal{Z}$].

Given a set K in $\mathbb{R}^d$, we say that K is *convex* if for every pair of points s and t in K, the line segment $\{(\lambda - 1)s + \lambda t \mid 0 \leq \lambda \leq 1\}$ connecting s to t lies entirely inside of K. Given a set G in $\mathbb{R}^d$, we define its *convex hull* to be the intersection of all convex sets K with $G \subseteq K$. That is,

$$\operatorname{conv}(G) = \bigcap \{K \subseteq \mathbb{R}^d \mid G \subseteq K,\ K \text{ convex}\}.$$

Theorem 1. *Given a finite set* $G = \{x_1, \dots, x_k\} \subset \mathbb{R}^n$,

$$\operatorname{conv}(G) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; \text{each } \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{1.1}$$

Proof. First we will show that the right-hand side of equation 1.1 is contained in the left-hand side. If $\lambda_j = 1$ for some j, then $\lambda_m = 0$ for each $m \neq j$. In this case $\sum \lambda_i x_i = x_j$ which is clearly contained in $\operatorname{conv}(G)$.

Now assume that each $\lambda_i < 1$, and note that

$$\lambda_1 x_1 + \cdots + \lambda_k x_k = (1 - \lambda_k)\left(\frac{\lambda_1}{1 - \lambda_k} x_1 + \cdots + \frac{\lambda_{k-1}}{1 - \lambda_k} x_{k-1}\right) + \lambda_k x_k.$$

Thus, $\sum_{i=1}^{k} \lambda_i x_i$ lies on the line segment connecting x_k to the point

$$\frac{\lambda_1}{1 - \lambda_k} x_1 + \cdots + \frac{\lambda_{k-1}}{1 - \lambda_k} x_{k-1}.$$

By convexity, it will suffice to show that this point lies inside of $\mathrm{conv}(G)$.

Now, if we note that

$$\frac{1}{1-\frac{\lambda_{k-1}}{1-\lambda_k}} = \frac{1-\lambda_k}{1-\lambda_k-\lambda_{k-1}}$$

and repeat the above process for the coefficient of x_{k-1}, we get

$$\begin{aligned}
&\tfrac{\lambda_1}{1-\lambda_k}x_1 + \cdots + \tfrac{\lambda_{k-1}}{1-\lambda_k}x_{k-1} \\
&\qquad = \left(1-\tfrac{\lambda_{k-1}}{1-\lambda_k}\right)\left(\tfrac{\lambda_1}{1-\lambda_k-\lambda_{k-1}}x_1 + \cdots + \tfrac{\lambda_{k-2}}{1-\lambda_k-\lambda_{k-1}}x_{k-2}\right) + \tfrac{\lambda_{k-1}}{1-\lambda_k}x_{k-1}.
\end{aligned}$$

Clearly, $0 \le \frac{\lambda_{k-1}}{1-\lambda_k} < 1$, so it will now suffice to show that

$$\left(\frac{\lambda_1}{1-\lambda_k-\lambda_{k-1}}x_1 + \cdots + \frac{\lambda_{k-2}}{1-\lambda_k-\lambda_{k-1}}x_{k-2}\right) \in \mathrm{conv}(G)\,.$$

Repeating this process, we see that it will suffice to show that $\frac{\lambda_1}{1-\lambda_k-\cdots-\lambda_2}x_1 = \frac{\lambda_1}{1-(1-\lambda_1)}x_1 = x_1 \in \mathrm{conv}(G)$, which is trivial. This proves that each point $\sum_{i=1}^k \lambda_i x_i \in \mathrm{conv}(G)$.

The containment in the other direction is clear if we note that the set in the right hand side of equation (1.1) is convex, which proves equality and completes the proof. □

We can now define a *polytope* to be the convex hull of a finite set of points in $\mathbb{R}^d$. An alternate definition of a polytope is the intersection of finitely many closed halfspaces in some $\mathbb{R}^d$ if the intersection is bounded. We will not prove here that these definitions are equivalent. For a finite point set G, the *polytope of* G is $P(G) := \mathrm{conv}(G)$.

Given a set $G = \{x_1, \ldots, x_k\}$, the *affine span* of the polytope $P(G)$ is defined to be

$$\mathrm{aff}(P(G)) = \left\{ \sum_{i=1}^k \lambda_i x_i \;\middle|\; \sum_{i=1}^k \lambda_i = 1 \right\}. \tag{1.2}$$

Notice that the only difference between this equation and equation (1.1) defining the convex hull of G is that the restriction that each $\lambda_i \ge 0$ has been removed. An *affine relation* on G is any k-tuple $(\lambda_1, \ldots, \lambda_k)$ such that

$$\sum_{i=1}^k \lambda_i x_i = 0 \quad \text{with} \ \sum_{i=1}^k \lambda_i = 0.$$

The *dimension* of a polytope $P(G)$ is the dimension of its affine span. Furthermore, if the number of independent affine relations on G (that is, the number of affine relations such that any affine relation on G is a linear combination of these affine relations) is r, then the dimension of the polytope is $|G| - r - 1$. We call a d-dimensional polytope a d-*polytope*.

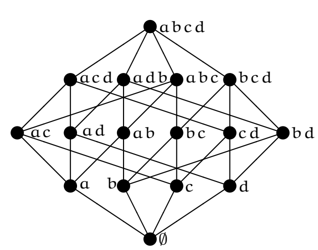
A *face* of a d-polytope $P(G)$ is any set of the form

$$F = P(G) \cap \{x \in \mathbb{R}^d \mid c \cdot x = c_0\}$$

where $c \in \mathbb{R}^d$ and $c_0 \in \mathbb{R}$ are chosen such that $c \cdot x \leq c_0$ for any x in $P(G)$. Intuitively, a face is the intersection of the polytope with any hyperplane that does not cut the polytope. An elementary result in the theory of polytopes states that any face of a polytope is itself a polytope. The *vertices*, *edges*, *ridges* and *facets* of a d-polytope are its 0-, 1-, $(d-2)$- and $(d-1)$-dimensional faces, respectively. The empty set $\emptyset$ is considered to be a (-1)-dimensional face. A face with dimension at least 0 and at most $d-1$ is called a *proper face*.

A d-*dimensional simplex*, or d-*simplex*, is a d-polytope with $d+1$ vertices. A d-polytope is called *simplicial* if each of its facets is a simplex—that is, each of its facets contains d vertices. A *simple* polytope, on the other hand, is a d-polytope for which each vertex is contained in d facets (the minimum number possible). It is simple to prove that every facet of a simplex is itself a simplex, and that therefore every proper face of a simplicial polytope is a simplex.

For any polytope P, we can construct a poset (*p*artially *o*rdered *set*) called the *face lattice* of P whose elements are the faces of P. The face lattice, denoted $\mathcal{L}(P)$, is partially ordered by inclusion of faces. Thus, given a d-dimensional face F of a polytope P, the faces directly 'above' F in $\mathcal{L}(P)$ have dimension $d+1$, and those directly 'below' F have dimension $d-1$. Furthermore, the polytope P is the unique 'highest' face in the lattice, and the empty set $\emptyset$ (the (-1)-dimensional face) is the unique 'lowest' face. Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic.



The face lattice of the three-dimensional simplex $P(\{a, b, c, d\})$

Given a polytope $P(G)$ with vertices $V(G) = \{x_1, \ldots, x_k\}$,

$$\text{relint}(P(G)) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; \text{each } \lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

is the *relative interior* of $P(G)$. The relative interior of a polytope is the convex hull of the polytope minus the points on its surface. Notice that the only difference between this equation and equation (1.1) is the tighter restriction that $\lambda_i > 0$.

Given two polytopes P_1 and P_2 we say that P_2 is the *combinatorial dual* of P_1 if $\mathcal{L}(P_1)$ is anti-isomorphic to $\mathcal{L}(P_2)$. That is, the two polytopes are dual if $\mathcal{L}(P_2)$ is just $\mathcal{L}(P_1)$ flipped upside-down. Furthermore, we can define the *polar dual* of $P \subset \mathbb{R}^d$ as

$$P^* = \{c \in \mathbb{R}^d \mid c \cdot x \leq 1 \text{ for all } x \in P\},$$

where it is assumed that the origin is contained in the relative interior of P. It is possible to define the polar dual for general polytopes, regardless of location of the origin, but for our purposes this assumption will be sufficient. [$\mathcal{Z}$] provides a good treatment of this construction, which associates with each vertex of P a facet of P^*, where the line connecting a vertex to the origin is normal to the resulting facet and the facet's distance from the origin is the multiplicative inverse of that of the vertex.

For a proof of the following proposition, we refer the reader to [$\mathcal{Z}$].

Proposition 2. *Given a polytope* P,

i) $\dim(P) = \dim(P)^*$,

ii) $P \cong P^{**}$ *(see page 8 for definition of '$\cong$')*,

iii) *If* $P = \text{conv}(V)$, *then* $P^* = \{p \mid p \cdot v \leq 1$ *for all* $v \in V\}$,

iv) *If* $P = \text{conv}(V)$, *then the facets of* P^* *are given by* $\{p \in P \mid v \cdot p = 1\}$ *for each* $v \in V$,

v) *If* $P \subset \mathbb{R}^n$, *and* $C \subset \mathbb{R}^n$ *is a set such that* $P = \{p \in \mathbb{R}^n \mid c \cdot p \leq 1\ \forall\, c \in C\}$, *then* $P^* = \text{conv}(C)$.

For the purposes of this thesis, we mainly care about the combinatorial structure and dimension of a polytope, not it's orientation, position or size. The following propositions show that affine transformations of a polytope — that is,

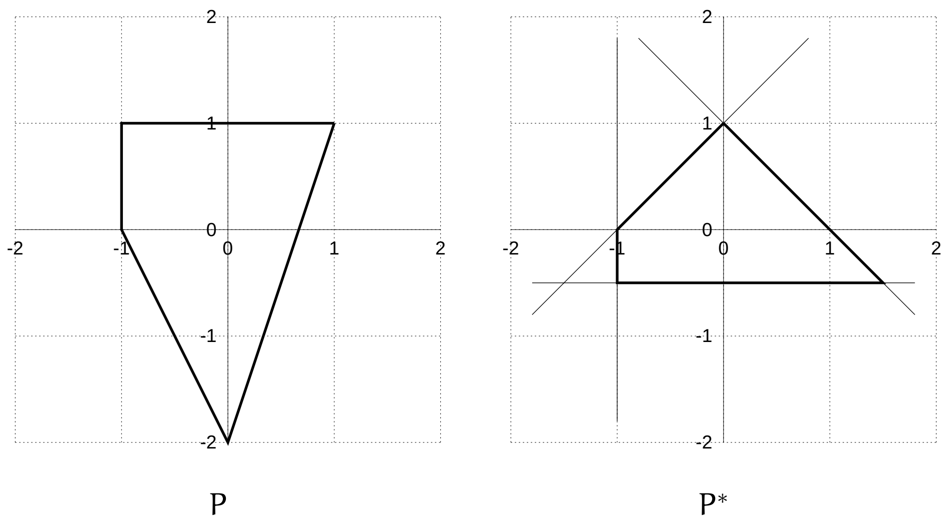


Figure 1.1: A polytope P and its polar dual P^*

transformations that consist of a linear transformation and a translation — don't change any of the qualities we care about.

Proposition 3. *Let* $P \subset \mathbb{R}^m$ *be a polytope, and* $A\colon \mathbb{R}^m \to \mathbb{R}^m$ *an invertible affine transformation. Then* $A(P)$ *is a polytope combinatorially equivalent to* P*.*

Proof. Let $A = T \circ L$, the composition of a linear transformation, L, and a translation, T, by $v \in \mathbb{R}^m$.

First, we will show that $A(P)$ is a polytope. Let $P = \text{conv}(\{x_1, \ldots, x_m\})$. Each $p \in P$, then, has the form $p = \sum \lambda_i x_i$. Thus, by linearity, $L(p) = \sum \lambda_i L(x_i)$. Noticing that $v = v \sum \lambda_i$ and applying the translation T, we get

$$A(p) = T \circ L(p) = v + \sum \lambda_i L(x_i) = \sum \lambda_i (v + L(x_i)) = \sum \lambda_i A(x_i),$$

which shows that $A(P) = \text{conv}(\{A(x_1), \ldots, A(x_m)\})$. Hence $A(P)$ is a polytope.

We will now show that $A(P)$ and P are combinatorially equivalent. An elementary result from linear algebra tells us that any linear function from $\mathbb{R}^m \to \mathbb{R}$ is equivalent to the dot product map $(a\cdot)\colon x \mapsto a \cdot x$ for some $a \in \mathbb{R}^m$. Let $F = \{p \in P \mid c \cdot p = c_0\}$ be a face of P. Then we have the map $(c\cdot)\colon p \mapsto c \cdot p = c_0$ for all $p \in F$. Additionally, $(c\cdot) \circ A^{-1}$ is a map from $\mathbb{R}^m \to \mathbb{R}$, and so there must exist some c' such that $(c\cdot) \circ A^{-1} = (c'\cdot)$. Thus,

$$(c'\cdot)\colon A(p) \mapsto c \cdot A^{-1}(A(p)) = c_0$$

and $c' \cdot A(p) = c_0$ for all $p \in F$. In addition, given *any* $p \in P$, we have $c \cdot p \leq c_0$. Therefore $c' \cdot A(p) = c \cdot p \leq c_0$, and $A(F) = \{A(p) \mid p \in F\} = \{p \in A(P) \mid c' \cdot p = c_0\}$ is a face of $A(P)$.

Clearly, the invertible affine transformation A is bijective and preserves subset inclusions, and the two polytopes must have isomorphic face lattices. Thus, $A(P)$ is combinatorially equivalent to P. □

If $P \subset \mathbb{R}^r$ and $Q \subset \mathbb{R}^s$ are polytopes, we say that P is *isomorphic* to Q if there exists an affine transformation $A \colon \mathbb{R}^r \to \mathbb{R}^r$ such that A is injective when restricted to P and $A(P) = Q$. In this case, we write $P \cong Q$. By proposition 3, if two polytopes are isomorphic, then they are combinatorially equivalent.

Notice that if $P \subset \mathbb{R}^m$ is an n-polytope with $n \leq m$, then we can choose an affine transformation $A \colon \mathbb{R}^m \to \mathbb{R}^m$ such that $A(\text{aff}(P)) = \mathbb{R}^n$. Thus A can be seen as a projection sending the n-polytope $P \subset \mathbb{R}^m$ to an isomorphic and full-dimensional n-polytope $A(P) \subset \mathbb{R}^n$. Thus in most cases we can assume without loss of generality that a polytope P of dimension n is full-dimensional, that is, $P \subset \mathbb{R}^n$.

1.2 Standard Constructions

In preparation for chapter three, we will define three different ways to combine polytopes: the *cartesian product*, the *direct sum*, and the *free join*.

Let $P_1 \subset \mathbb{R}^r$ be an r-polytope and $P_2 \subset \mathbb{R}^s$ be an s-polytope. We define the *cartesian product* of P_1 and P_2 to be the polytope

$$P_1 \times P_2 = \text{conv}(\{(x, y) \in \mathbb{R}^{r+s} \mid x \in P_1, y \in P_2\})\,.$$

If $\vec{0} \in \text{relint}(P_1)$ and $\vec{0} \in \text{relint}(P_2)$, then the polytope

$$P_1 \oplus P_2 = \text{conv}\Big(\{(x, \vec{0}_s) \in \mathbb{R}^{r+s} \mid x \in P_1\} \,\cup\, \{(\vec{0}_r, y) \in \mathbb{R}^{r+s} \mid y \in P_2\}\Big)$$

defines the *direct sum* (or *free sum*) of P_1 and P_2. Finally, let $P_1 \subset \mathbb{R}^{r+s+1}$ be an r-polytope and $P_2 \subset \mathbb{R}^{r+s+1}$ be an s-polytope. If $\text{aff}(P_1)$ and $\text{aff}(P_2)$ are *skew* — that is, they do not intersect and contain no parallel lines — then the *free join* of the polytopes is

$$P_1 \bowtie P_2 = \text{conv}(P_1 \cup P_2)\,.$$

Some simple visual examples of these operations are useful.

Example 4. Let $P_1 = \operatorname{conv}((1),(3)) \subset \mathbb{R}$, a line segment of length 2, and let $P_2 = \operatorname{conv}((-1,-1),(1,-1),(1,1),(-1,1)) \subset \mathbb{R}^2$, a two-by-two square. We construct the cartesian product

$$P_1 \times P_2 = \operatorname{conv}\begin{pmatrix} (1,-1,-1), & (1,1,-1), & (1,1,1), & (1,-1,1), \\ (3,-1,-1), & (3,1,-1), & (3,1,1), & (3,-1,1) \end{pmatrix},$$

a two-by-two-by-two cube in $\mathbb{R}^3$. (See figure 1.2.) △

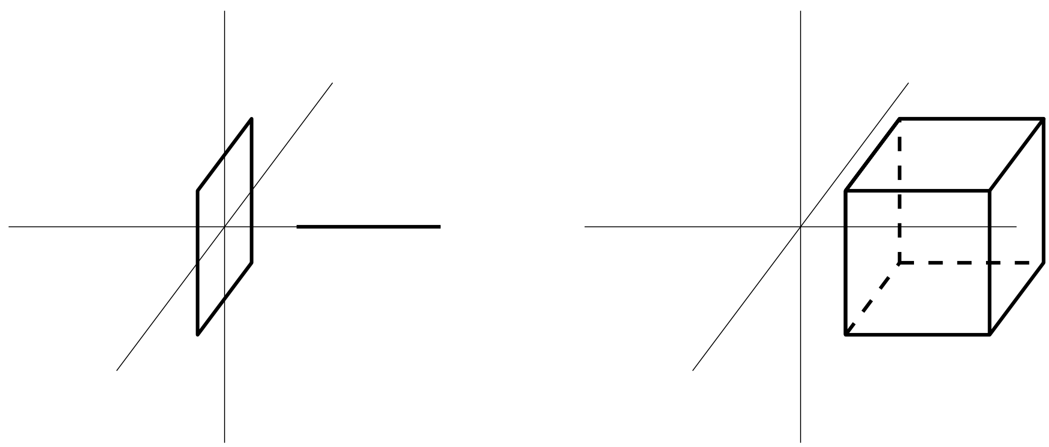


Figure 1.2: Cartesian product

Example 5. Let P_2 be defined as in example 4, but let $P_1 = \operatorname{conv}((-1),(1)) \subset \mathbb{R}$ so that now the origin is in the relative interior of both polytopes. We can now construct the direct sum

$$P_1 \oplus P_2 = \operatorname{conv}\begin{pmatrix} \{(0,-1,-1),(0,1,-1),(0,1,1),(0,-1,1)\} \\ \cup \ \{(-1,0,0),(1,0,0)\} \end{pmatrix},$$

a bipyramid in $\mathbb{R}^3$. (See figure 1.3.) △

Example 6. Unfortunately, the free join of a square and a line is four-dimensional, so we will have to simplify the setup of the previous two examples a bit to be able to visualize it. Let $P_1 = \operatorname{conv}((0,-1,1),(0,1,1))$ and let $P_2 = \operatorname{conv}((-1,0,0),(1,0,0))$. The affine spans of these polytopes are clearly skew, allowing us to take the free join

$$P_1 \bowtie P_2 = \operatorname{conv}(\{(0,-1,1),(0,1,1)\} \ \cup \ \{(-1,0,0),(1,0,0)\}),$$

a simplex in $\mathbb{R}^3$. (See figure 1.4.) △

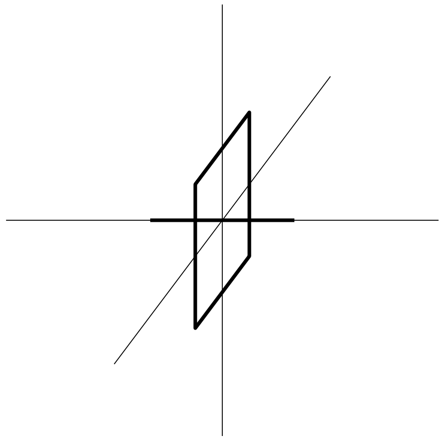


Figure 1.3: Direct sum

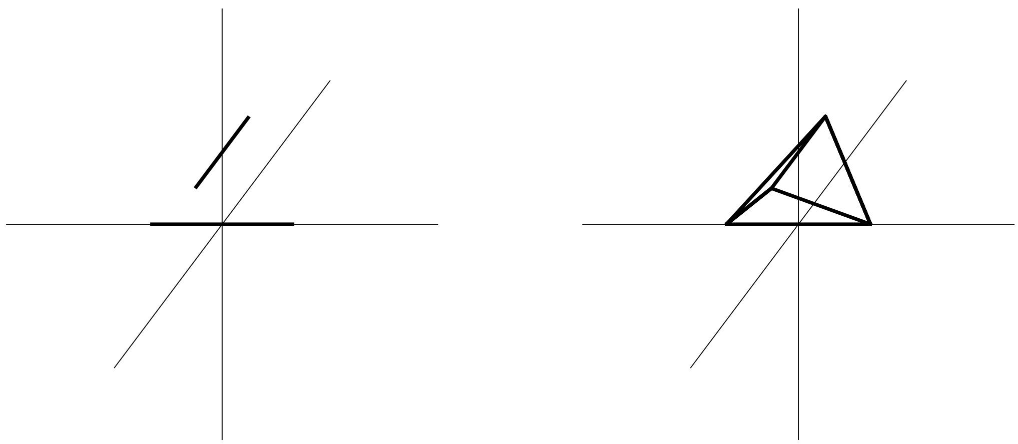


Figure 1.4: Free join

The following few propositions characterize the face lattices of the polytopes resulting from these three operations.

Proposition 7. *Let P_1 and P_2 be polytopes. The k-faces of $P_1 \times P_2$, for $k \geq 0$, are precisely $F_1 \times F_2$ where F_1 is a nonempty i-face of P_1, F_2 is a nonempty j-face of P_2, and $i + j = k$.*

Proof. First we will prove that, given F_1 and F_2, $F_1 \times F_2$ is a face of $P_1 \times P_2$. Let F_i be defined by the equation $c_i \cdot x_i = \alpha_i$, where $c_1 \in \mathbb{R}^r$, $c_2 \in \mathbb{R}^s$, and $\alpha_i \in \mathbb{R}$. Define $c = (c_1, c_2) \in \mathbb{R}^{r+s}$ and $\alpha = \alpha_1 + \alpha_2$. Then $c \cdot p \leq \alpha$ for all $p \in P_1 \times P_2$, with equality only if $p \in F_i$.

To prove the converse, let F be a nonempty face of $P_1 \times P_2$ defined by $c \cdot (x, y) = \alpha$ for some $c \in \mathbb{R}^{r+s}$ and some $\alpha \in \mathbb{R}$. Let $c_1 \in \mathbb{R}^r$ and $c_2 \in \mathbb{R}^s$ be defined by $c = (c_1, c_2)$. We define $\alpha_i = \max\{c_i \cdot x_i \mid x_i \in P_i\}$, so that $c_i \cdot x_i = \alpha_i$ for at least one

$x_i \in P_i$. That is,

$$c_1 \cdot x_1 + c_2 \cdot x_2 = \alpha_1 + \alpha_2 = \alpha$$

for at least one pair $(x_1, x_2) \in P_1 \times P_2$. But then, $c_i \cdot x_i = \alpha_i$ must hold for all $(x_1, x_2) \in F$. Thus, we identify the faces $F_i = \{x_i \in P_i \mid c_i \cdot x_i = \alpha_i\} \subseteq P_i$, and $F = F_1 \times F_2$, which completes the proof. □

Proposition 8. *Let* P_1 *and* P_2 *be polytopes. The* k*-faces of* $P_1 \bowtie P_2$ *are precisely* $F_1 \bowtie F_2$ *where* $\dim(F_1) + \dim(F_2) + 1 = k$.

Proof. We will use an affine transformation to embed the polytopes into more manageable spaces. Assume $P_1 \subset \mathbb{R}^r$ and $P_2 \subset \mathbb{R}^s$ are embedded in $\mathbb{R}^{r+s+1}$. A simple linear change of coordinates allows us to let $\text{aff}(P_1) = p + \text{span}(e_1, \dots, e_r)$ and $\text{aff}(P_2) = q + \text{span}(e_{r+1}, \dots, e_{r+s})$ where $p = (p_1, \dots, p_{r+s+1})$ and $q = (q_1, \dots, q_{r+s+1})$ are vectors in $\mathbb{R}^{r+s+1}$, and $\{e_1, \dots, e_{r+s+1}\}$ are the standard basis vectors. Translating the polytopes by $-(q_1, \dots, q_r, p_{r+1}, \dots, p_{r+s+1})$ gives us

$$\begin{aligned} \text{aff}(P_1) = \text{span}(e_1, \dots, e_r) &= \mathbb{R}^r \times \{\vec{0}_s\} \times \{0\} \\ \text{aff}(P_2) = \alpha e_{r+s+1} + \text{span}(e_{r+1}, \dots, e_{r+s}) &= \{\vec{0}_r\} \times \mathbb{R}^s \times \{\alpha\} \end{aligned}$$

for some $\alpha \in \mathbb{R}$. Now, notice that $\alpha \neq 0$ because otherwise $\vec{0}_{r+s+1} \in \text{aff}(P_1) \cap \text{aff}(P_2)$, and the affine spans have a point of intersection. We can thus normalize to set $\alpha = 1$ with a linear map. Therefore, there is an affine transformation that will transform two skew polytopes P_1 and P_2 into isomorphic polytopes

$$\begin{aligned} P_1 &\subset \mathbb{R}^r \times \{\vec{0}_s\} \times \{0\} \\ P_2 &\subset \{\vec{0}_r\} \times \mathbb{R}^s \times \{1\}. \end{aligned}$$

Let F_i be a nonempty face of P_i with defining equation

$$c_i \cdot x = (c_{i,1}, \dots, c_{i,r+s+1}) \cdot x = \alpha_i,$$

for $i = 1, 2$. Notice that c_1 (resp. c_2) can be modified by setting $c_{1,r+1}, \dots, c_{1,r+s}$ (resp. $c_{2,1}, \dots, c_{2,r}$) equal to zero without affecting the face. In fact, by adjusting c_2, we can assume $\alpha_1 = \alpha_2$. To see this, notice that adding some constant γ to both α_2 and $c_{2,r+s+1}$ leaves the hyperplane unchanged, since the last coordinate of every point in P_2 is 1. Let $\gamma = \alpha_1 - \alpha_2$, so that c_2 becomes $c_2' = c_2 + \gamma e_{r+s+1}$ and α_2 becomes α_1. Then the equation $c_2' \cdot x = \alpha_1$ defines F_2. Defining $c = c_1 + c_2'$, the equation $c \cdot x = \alpha_1$ holds for all $x \in F_1 \bowtie F_2$. Furthermore, if $p \in P_1 \times P_2$, then $p = \sum_{x_i \in P_1 \cup P_2} \lambda_i x_i$ with $\sum \lambda_i = 1$. Hence,

$$c \cdot p = \sum \lambda_i (c \cdot x_i) \leq \alpha \sum \lambda_i = \alpha,$$

and $F_1 \bowtie F_2$ is a face of $P_1 \bowtie P_2$. Therefore the join of any two nonempty faces of the polytopes is a face of the join.

If at least one of the F_i — say F_2 — is empty, then we have $F_1 \bowtie \emptyset = F_1$. Let $(c_1, \ldots, c_{r+s+1}) \cdot x = \alpha$ be a defining equation for F_1. If we define $c' = (c_1, \ldots, c_r, 0, \ldots, 0)$, then $c' \cdot x = \alpha$ for all $x \in F_1 \times \emptyset$. Also, by the argument given in the previous paragraph, $c' \cdot p \leq \alpha$ for $p \in P_1 \bowtie P_2$. Thus the join of any two faces, empty or not, defines a face of $P_1 \bowtie P_2$.

We will now show that every face of $P_1 \bowtie P_2$ is $F_1 \bowtie F_2$ for some faces F_1 of P_1 and F_2 of P_2. Let F be a face of $P_1 \bowtie P_2$ defined by the equation $c \cdot x = (c_1, c_2, 0) \cdot x = \alpha$. The assumption that the last coordinate of c is zero is made without loss of generality by the argument in the previous paragraphs. Each point $x \in F$ must be of the form $x = tp + (1-t)q$ with $p \in P_1$ and $q \in P_2$. If $t = 0$ or $t = 1$ in this formula, then $x \in P_i$ for some i, and is thus on the face $P_1 \bowtie \emptyset$ or $\emptyset \bowtie P_2$. If, however, t is not zero or one, then the entire line segment $\{tp + (1-t)q \mid 0 \leq t \leq 1\}$ must be contained in F, and

$$c \cdot (tp + (1-t)q) = tc_1 \cdot p + (1-t)c_2 \cdot q = \alpha$$

for all $0 \leq t \leq 1$. Therefore, $c_1 \cdot p = c_2 \cdot q = \alpha$, which defines a face F_1 of P_1 and a face F_2 of P_2. Hence, F must be equal to $F_1 \bowtie F_2$, which completes the proof. □

The following theorem relates the operations of cartesian product and direct sum through the polar dual.

Theorem 9. *Given polytopes* $P_1 \subset \mathbb{R}^r$ *and* $P_2 \subset \mathbb{R}^s$, *if the origin* $\vec{0} \in \mathrm{relint}(P_1)$ *and* $\vec{0} \in \mathrm{relint}(P_2)$, *then* $(P_1^* \times P_2^*)^* = P_1 \oplus P_2$.

Proof. We give Bremner's proof [Br].

Let $P_i = \mathrm{conv}(V_i)$. Proposition 2 says that $P_i^* = \{p_i \mid p_i \cdot v_i \leq 1 \text{ for all } v_i \in V_i\}$, and that the facets of P_i^* are given by $\{p_i \mid v_i \cdot p_i = 1\}$ for each $v_i \in V_i$. By proposition 7, the facets of $P_1^* \times P_2^*$ are given by $P_1^* \times F_2$ and $F_1 \times P_2^*$, where F_i is a facet of P_i^*. Let P_i^* be defined by the equation $\vec{0} \cdot p = 0$. Using the construction for proposition 7 the facets of $P_1^* \times P_2^*$ are defined by the equations $(v_1, \vec{0}_s) \cdot p = 1$ and

$(\vec{0}_r, v_2) \cdot p = 1$, where $v_i \in V_i$, and $P_1^* \times P_2^*$ is defined by the halfspaces

$$\{p \in \mathbb{R}^{r+s} \mid (v_1 \times \vec{0}_s) \cdot p \leq 1 \text{ and } (\vec{0}_r \times v_2) \cdot p \leq 1\}$$

for $v_i \in V_i$. If we apply proposition 2(v) to $P_1^* \times P_2^*$, then we get

$$(P_1^* \times P_2^*)^* = \operatorname{conv}\Big(\{(v_1, \vec{0}_s) \in \mathbb{R}^{r+s} \mid v_1 \in V_1\} \cup \{(\vec{0}_r, v_2) \in \mathbb{R}^{r+s} \mid v_2 \in V_2\}\Big),$$

which is precisely the definition of the direct sum, $P_1 \oplus P_2$. □

The cartesian product ($\times$) of two face lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ is a poset by the following rule: $(x_1, y_1) \leq (x_2, y_2)$ in $\mathcal{L}_1 \times \mathcal{L}_2$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. We are now ready for the following theorem, which fully characterizes the structure of $\mathcal{L}(P_1 \times P_2)$, $\mathcal{L}(P_1 \oplus P_2)$, and $\mathcal{L}(P_1 \bowtie P_2)$.

Theorem 10. *Let* P_1 *and* P_2 *be polytopes with face lattices* $\mathcal{L}(P_1)$ *and* $\mathcal{L}(P_2)$*. Then*

i) $\mathcal{L}(P_1 \times P_2) \cong \big(\mathcal{L}(P_1) \times \mathcal{L}(P_2)\big)/(\sim_\emptyset)$

ii) $\mathcal{L}(P_1 \oplus P_2) \cong \big(\mathcal{L}(P_1) \times \mathcal{L}(P_2)\big)/(\sim_P)$

iii) $\mathcal{L}(P_1 \bowtie P_2) \cong \mathcal{L}(P_1) \times \mathcal{L}(P_2)$,

where $\sim_\emptyset$ *is the equivalence relation defined by* $(F \times \emptyset) \sim_\emptyset (\emptyset \times F) \sim_\emptyset (\emptyset \times \emptyset)$ *and* $\sim_P$ *is the equivalence relation defined by* $(F \times P_2) \sim_P (P_1 \times F) \sim_P (P_1 \times P_2)$ *for all faces* F.

Thus, $\mathcal{L}(P_1 \times P_2)$ is constructed from $\mathcal{L}(P_1) \times \mathcal{L}(P_2)$ by identifying all of the elements of the form $\emptyset \times F_2$ or $F_1 \times \emptyset$. Similarly, $\mathcal{L}(P_1 \oplus P_2)$ is constructed from $\mathcal{L}(P_1) \times \mathcal{L}(P_2)$ by identifying all of the elements of the form $P_1 \times F_2$ or $F_1 \times P_2$. Example 11 will be helpful in understanding this theorem and its proof.

Proof.

i) By proposition 7, we know that every face of $P_1 \times P_2$ is the cartesian product $F_1 \times F_2$, where F_i is a face of P_i. If $F_1 = \emptyset$ or $F_2 = \emptyset$, then $F_1 \times F_2 = \emptyset$. Therefore, we can construct an obvious isomorphism between $\mathcal{L}(P_1 \times P_2)$ and $\big(\mathcal{L}(P_1) \times \mathcal{L}(P_2)\big)/(\sim_\emptyset)$.

ii) Theorem 9 gives us $\mathcal{L}(P_1 \oplus P_2) = \mathcal{L}((P_1^* \times P_2^*)^*)$. If we apply part (i) and recall the relation between the face lattice of a polytope and its dual (page 6), our result follows directly.

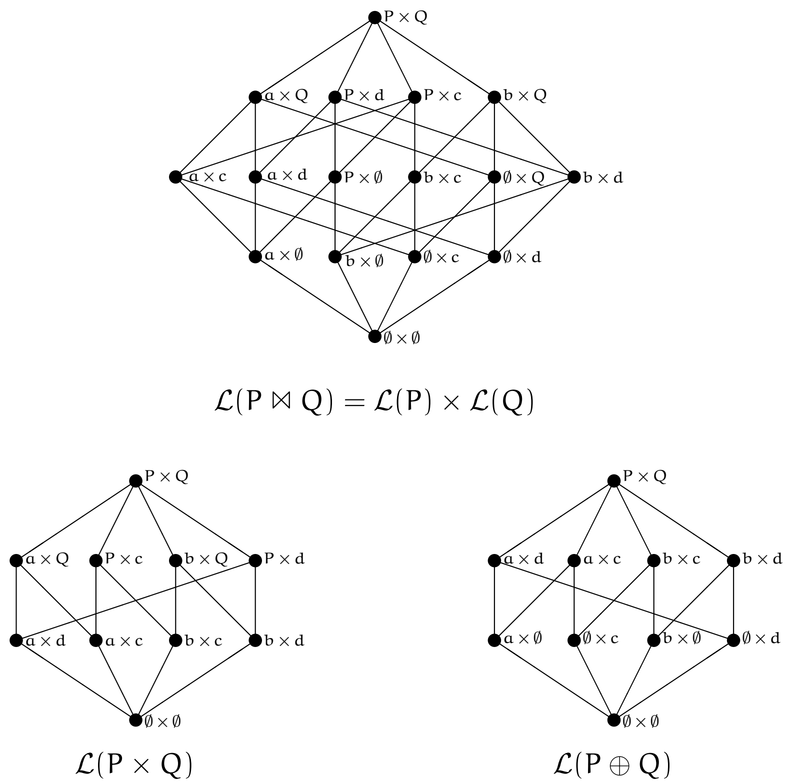


Figure 1.5: Face lattices of $P \bowtie Q$, $P \times Q$ and $P \oplus Q$

iii) Using proposition 8, the isomorphism is just $F_1 \bowtie F_2 \mapsto F_1 \times F_2$. □

Example 11. Let P be the line segment $\text{conv}(a, b)$ and let Q be the line segment $\text{conv}(c, d)$. The face lattices of $P \bowtie Q$, $P \times Q$, and $P \oplus Q$ are shown in figure 1.5. It is assumed that $\text{aff}(P)$ and $\text{aff}(Q)$ are skew in the case of $P \bowtie Q$, and complementary in the case of $P \oplus Q$. △

Chapter 2

Group Polytopes

The aim of the study of group polytopes is to investigate the relationships between the structure of groups and the structure of their associated polytopes. This chapter will cover some of the basic general results, some of which will apply to our treatment of wreath products in the next chapter.

2.1 Getting Polytopes from Groups

So far, we have only spoken of polytopes of generic point sets. The subject of this thesis, though, is polytopes associated with groups. Before discussing how these are constructed we will need a bit of real representation theory. Recall that $GL(m, \mathbb{R})$ is the set of invertible $m \times m$ matrices over the real numbers.

Definition 12. Given a group G, a mapping $\rho\colon G \to GL(m, \mathbb{R})$ is a *real matrix representation* if $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in G$. That is, ρ is a representation if it is a group homomorphism from G to $GL(m, \mathbb{R})$. $\triangledown$

Although a representation of a group respects the structure of the group itself, structure can be lost through a representation, as evidenced by the trivial representation $\rho\colon G \to GL(1, \mathbb{R})$, where $\rho(g) = [1]$ for all g in G. A representation is said to be *faithful* if the homomorphism ρ is an injection, and thus loses none of the group's information in the mapping.

A common and useful kind of group representation is the permutation representation:

Definition 13. If $G \subseteq S_m$ is a permutation group of degree m then its *permutation representation*, $\rho_P\colon G \to GL(m, \mathbb{R})$, is defined as follows. If x is an element in G, then the (i, j)-th entry of $\rho_P(g)$ is given by

$$\rho_P(g)_{i,j} = \begin{cases} 1 & \text{if } g \text{ sends } j \text{ to } i \\ 0 & \text{otherwise.} \end{cases}$$

$\triangledown$

If $g \in G \subseteq S_m$, then $\rho_P(g)$ is an $m \times m$ matrix with m ones and $m^2 - m$ zeros. Furthermore, each row and each column contains exactly one nonzero entry.

Now, given a group element $g \in G$ and a representation $\rho\colon G \to GL(m, \mathbb{R})$, we can construct a point $\rho(g) \in \mathbb{R}^{m^2}$ by "flattening" the matrix $\rho(g)$. That is, create an m^2-vector by placing each successive row of $\rho(g)$ after the previous. In this way we can translate G into a point set $\rho(G) \subset \mathbb{R}^{m^2}$.

Definition 14. We can now, given a group G and a matrix representation ρ, define the *polytope of* G to be $P(G, \rho) = \text{conv}(\rho(G))$. $\triangledown$

Example 15. Let $G = C_3 = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$. Then the permutation representation $\rho_P(G)$ is $\left\{ \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} \right\}$. The flattened points in $\mathbb{R}^9$ are

$$(1,0,0,0,1,0,0,0,1), (0,0,1,1,0,0,0,1,0), (0,1,0,0,0,1,1,0,0).$$

So the polytope $P(G, \rho_P)$ is a two-dimensional simplex (a triangle) embedded in nine-dimensional space. $\triangle$

Example 16. Let G be as in the above example, but let ρ be the representation mapping G to $\left\{ \begin{bmatrix} 1&0\\0&1 \end{bmatrix}, \begin{bmatrix} -1&1\\-1&0 \end{bmatrix}, \begin{bmatrix} 0&-1\\1&-1 \end{bmatrix} \right\}$. So $P(G, \rho)$ is again a two-dimensional simplex, but it is now embedded in $\mathbb{R}^4$. $\triangle$

Notice that in both of these examples the representation of $G = C_3$ is faithful, but that the resulting polytopes differ.

2.2 Useful Theorems and Definitions

It would be nice to know which elements of a group G become vertices of $P(G)$ and which become internal points. As it turns out, they must all be vertices.

Theorem 17. *Let G be a group and $\rho\colon G \to GL(n, \mathbb{R})$ be any representation of G, giving us the polytope $P(G) = P(G, \rho)$. If h is an element of G, then $\rho(h)$ is a vertex of $P(G)$.*

Proof. Because $P = \operatorname{conv}(\{\rho(g) \mid g \in G\})$, the point $\rho(g)$ must be a vertex of P for some $g \in G$. Given some $h \in G$, construct an invertible linear transformation $hg^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as matrix multiplication by $\rho(hg^{-1})$. Thus, hg^{-1} sends $\rho(g)$ to $\rho(h)$. By proposition 3, $\rho(h)$ must be a vertex of $P(hG) = P(G)$. □

Edges of a group polytope are much harder to characterize in general than the vertices. Brualdi's theorem is a useful tool in characterizing edges.

Theorem 18. (Brualdi's Theorem) *Let $G \subseteq S_m$ be a permutation group and let $P(G)$ be the polytope associated with its permutation representation, ρ. Let $\rho(h), \rho(g) \in P(G)$ be two vertices. The line segment connecting $\rho(g)$ to $\rho(h)$ is an edge of the polytope if and only if the cycle decomposition of $g^{-1}h$ cannot be factored into two disjoint nontrivial parts, both of which are elements of G.*

Proof. For simplicity, we will identify the permutation group G with its permutation representation $\rho(G)$.

It will suffice to show that the theorem holds for the vertices e and g, where e is the identity permutation. In this case $e^{-1}g = g$, so we need to show that the line segment is an edge if and only if the cycle decomposition for g cannot be factored into a product of disjoint group elements. We will prove the forward direction first.

Suppose the cycle decomposition of g factors as $g = g_1g_2$, with $g_1, g_2 \in G$. Then, by swapping out the appropriate rows in the permutation matrix, we have $e + g = g_1 + g_2$ and therefore $\frac{1}{2}e + \frac{1}{2}g = \frac{1}{2}g_1 + \frac{1}{2}g_2$. This means that the point $\frac{1}{2}g_1 + \frac{1}{2}g_2$ lies on the line segment connecting e to g. But an elementary result from geometry tells us that two vertices of a convex polytope determine an edge if and only if no point on the line segment connecting the vertices can be represented as a nontrivial convex combination of two points of the polytope, at least one of which does not lie on the line segment. Therefore, if g factors as above, then the point $\frac{1}{2}g_1 + \frac{1}{2}g_2$ is a nontrivial combination of g_1 and g_2 and the segment connecting e to g cannot be an edge.

To prove the converse, we will show that if the line segment between e and g is not an edge, then the cycle decomposition of g factors nontrivially as $g = g_1g_2$ with $g_1, g_2 \in G$.

Assuming that the line segment connecting e to g is not an edge, let $x = \frac{1}{2}e + \frac{1}{2}g$ so that x lies on this line segment. Since x does not lie on an edge, it must be a positive convex combination of the other group elements:

$$x = \sum_{g_j \in G} \lambda_j g_j,$$

where each $\lambda_l \geq 0$, $\sum_l \lambda_l = 1$, and where the coefficient is nonzero for some g' not on the line segment. Fix any such g'. Because the above sum is nonnegative, any entry with a zero in the matrix x forces a zero in the corresponding entry of g'. Furthermore, because x is a sum of e and g, we must have either $g'(i) = i$ or $g'(i) = g(i)$.

Since g' does not lie on the line segment connecting e to g, there must be some i_1 for which $g'(i_1) \neq i_1$. Therefore $g'(i_1) = g(i_1)$, or, letting $(i_1, \ldots, i_m)$ be a cycle in the cycle decomposition of g, $g'(i_1) = i_2$. Now, assuming $g'(i_k) = g(i_k) = i_{k+1}$ for some $k < m$, we know that either $g'(i_{k+1}) = i_{k+1}$ or $g'(i_{k+1}) = g(i_{k+1}) = i_{k+2}$. In the first case, we have $g'(i_k) = g'(i_{k+1})$, which is impossible because $i_k \neq i_{k+1}$. Thus the second case, $g'(i_{k+1}) = g(i_{k+1}) = i_{k+2}$, must be true, and we have shown by induction that both g and g' contain the cycle $(i_1, \ldots, i_m)$. This process can be repeated until all cycles of g' have been covered, showing that the cycle decomposition of g contains each disjoint cycle in the cycle decomposition of g'.

Now, gg'^{-1} cannot be the identity, because $g \neq g'$. Thus, $g = g'g''$ for some $g', g'' \in G$ whose cycles are disjoint, which completes the proof. □

2.3 Computing Polytopes

Using computers, it is possible to generate and analyze specific examples of polytopes associated with groups. Using GAP [GAP] for the general algebra, and PORTA [PORTA] for the polytope-specific tasks, it is possible to get complete information about almost any polytope (provided enough running time).

The following GAP script will take the j-th transitive group of order i (see [CHM]) and output the resulting point set to a file `polytope.poi`.

```
G:=TransitiveGroup(i,j);
n:=NrMovedPoints(G);
points:=List(Elements(G),x->Flat(PermutationMat(x,n)));
name:="polytope.poi";
AppendTo(name,"DIM=",n^2,"\n\n");
AppendTo(name,"COMMENT ",G," ",Size(G),"\n\n");
```

```
AppendTo(name,"CONV_SECTION\n");
for x in points do
  for y in x do
    AppendTo(name,y," ");
  od;
  AppendTo(name,"\n");
od;
AppendTo(name,"\n","END\n\n");
```

Then, using the `traf -v polytope.poi` command with PORTA, a file titled `polytope.poi.ieq` is generated. This file contains a list of the inequalities that define the facets of the polytope, and how many vertices lie on each of these facets.

For example, if we let the group in question be the symmetric group S_3, a relatively simple group (`TransitiveGroup(3,1)`), the resulting point set file, `polytope.poi`, is

```
DIM=9

COMMENT S3 6

CONV_SECTION
1 0 0 0 1 0 0 0 1
1 0 0 0 0 1 0 1 0
0 1 0 1 0 0 0 0 1
0 1 0 0 0 1 1 0 0
0 0 1 1 0 0 0 1 0
0 0 1 0 1 0 1 0 0

END
```

The corresponding inequalities file, `polytope.poi.ieq`, is

```
DIM = 9

VALID
0 0 1 0 1 0 1 0 0
```

```
INEQUALITIES_SECTION
(  1)       +x3      +x6-x7-x8    == 0
(  2)    +x2+x3-x4      -x7       == 0
(  3)          +x4+x5+x6-x7-x8-x9 == 0
(  4) +x1+x2+x3-x4-x5-x6          == 0
(  5)                   +x7+x8+x9 == 1

(  1) -x5-x6-x8-x9 <= -1
(  2) -x5          <=  0
(  3)    -x6       <=  0
(  4)       -x8    <=  0
(  5)          -x9 <=  0
(  6)       +x8+x9 <=  1
(  7)    +x6   +x9 <=  1
(  8) +x5   +x8    <=  1
(  9) +x5+x6       <=  1

END

strong validity table :
\ P      |         |
 \ O     |         |
I \ I    |         |
 N \ N   | 1     6 | #
  E \ T  |         |
   Q \ S |         |
    S \  |         |
       \ |         |
----------------------
1        | ..*** * :  4
2        | .**** . :  4
3        | *.*.* * :  4
4        | *.**. * :  4
5        | .*.** * :  4
6        | ***.* . :  4
7        | ****. . :  4
8        | **..* * :  4
```

```
9          | **.*. * :  4
            .............
#          | 66666 6
```

From this we can see that $\mathrm{P}(S_3)$ has nine facets, each containing four of the six vertices. In addition, the inequalities defining the facets are given.

Table 2.1 provides a list of transitive permutation groups up to degree seven. The groups are organized by degree and traditional ordering (see [CHM]), and are listed with name, size, number of facets, dimension, size of the 1-stabilizer, and the number of vertices per facet. The blank fields required unreasonable computing times, and were therefore ignored.

deg.	num.	name	size	# facets	dim.	$\|G_1\|$	# verts./fac.
2	1	S_2	2	2	1	1	1
3	1	A_3	3	3	2	1	2
3	2	S_3	6	9	4	2	4
4	1	$C(4) = 4$	4	4	3	1	3
4	2	$E(4) = 2[\times]2$	4	4	3	1	3
4	3	$D(4)$	8	8	5	2	6
4	4	A_4	12	64	9	3	9
4	5	S_4	24	16	9	6	18
5	1	$C(5) = 5$	5	5	4	1	4
5	2	$D(5) = 5:2$	10	25	8	2	8
5	3	$F(5) = 5:4$	20	625	16	4	16
5	4	A_5	60	8665	16	12	16, 18, 19, 20, 48
5	5	S_5	120	25	16	24	96
6	1	$C(6) = 6$	6	6	5	1	5
6	2	$D_6(6)$	6	6	5	1	5
6	3	$D(6) = S(3)[\times]2$	12	18	9	2	10
6	4	$A_4(6) = [2^2]3$	12	12	11	2	11
6	5	$F_{18}(6) = 3 \wr 2$	18	12	9	3	15
6	6	$2A_4(6) = 2 \wr 3$	24	18	11	4	20
6	7	$S_4(6d)$	24	82	13	4	13, 18, 20
6	8	$S_4(6c)$	24	474	13	4	13, 14, 20
6	9	$F_{18}(6):2$	36	72	17	6	30
6	10	$F_{36}(6)$	36	72	17	6	30
6	11	$2S_4(6) = 2 \wr S(3)$	48	18	13	8	40
6	12	$L(6) = A_5(6)$	60	1334581	25	10	
6	13	$F_{36}(6):2 = S(3) \wr 2$	72	36	17	12	60
6	14	$L(6):2 = S_5(6)$	120		25	20	
6	15	A_6	360		25	60	
6	16	S_6	720	36	25	120	600
7	1	$C(7) = 7$	7	7	6	1	6
7	2	$D(7) = 7:2$	14	49	12	2	12
7	3	$F_{21}(7) = 7:3$	21	343	18	3	18
7	4	$F_{42}(7) = 7:6$	42	117649	36	6	36
7	5	$L(7) = L(3,2)$	168		36	24	
7	6	A_7	2520		36	360	
7	7	S_7	5040		36	720	

Table 2.1: Transitive Permutation Polytopes

Chapter 3

Wreath-Product Polytopes

The goal of this chapter will be to completely characterize polytopes of groups arising as the wreath product of two groups, one of which is a regular permutation group. First, though, we will need to define wreath products and derive some basic results about them.

3.1 Wreath Products of Groups

Given a group G and a permutation group $H \subseteq S_n$, the *wreath product* $G \wr H = \{(g, h) \mid g \in G^n, h \in H\}$ is a group under the operation defined by

$$(g', h')(g, h) = ((g'_1, \ldots, g'_n), h')((g_1, \ldots, g_n), h) := ((g'_{h(1)}g_1, \ldots, g'_{h(n)}g_n), h'h). \tag{3.1}$$

Clearly, $|G \wr H| = |H||G|^{|H|}$.

Because this rule for multiplication is cumbersome in practice, we will use the following notation when working with wreath products. Any element $(g, h) = ((g_1, \ldots, g_n), h)$ of $G \wr H$ can be represented by hI_g where I_g is the diagonal matrix with $\{g_1, \ldots, g_n\}$ on the diagonal. Equivalently, the element can be represented by the $n \times n$ permutation matrix corresponding to h, but with g_i replacing the 1 in the i-th column ($i = 1, \ldots, n$).

Example 19. For example, let $G = \{e, g\}$, and let $H = S_2 \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Then

$$G \wr H \cong \left\{ \begin{matrix} \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}, & \begin{bmatrix} g & 0 \\ 0 & e \end{bmatrix}, & \begin{bmatrix} e & 0 \\ 0 & g \end{bmatrix}, & \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}, \\ \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}, & \begin{bmatrix} 0 & e \\ g & 0 \end{bmatrix}, & \begin{bmatrix} 0 & g \\ e & 0 \end{bmatrix}, & \begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix} \end{matrix} \right\}. \tag{3.2}$$

△

If G is also a permutation group, say $G \subseteq S_m$, then $G \wr H$ is a permutation subgroup of S_{mn}. In this case we have a permutation representation for the group. The construction of the permutation representation is similar to that of the generic wreath product, except that each element of an $n \times n$ permutation matrix is changed to an $m \times m$ block representing an element of G. Thus each zero is converted to a block of m^2 zeros. This yields an $mn \times mn$ matrix.

Example 20. If we let $G' = H = S_2 \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, then the permutation representation of $G' \wr H$ is

$$\left\{ \begin{array}{l} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \right\}.$$

This should be compared to display (3.2) in the above example, noting that the two groups G and G' are isomorphic. △

We will now devote some attention to the structure of the group $G \wr H$. Let G be any finite group, and let H be a *regular* subgroup of S_n. (That is, the identity is the only element of H that has a fixed point.) Then we can write the wreath product as

$$G \wr H = \left\{ h \begin{bmatrix} g_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_n \end{bmatrix} \;\middle|\; h \in H;\; g_1, \ldots, g_n \in G \right\}.$$

Call the subgroup of $G \wr H$ defined by

$$(G \wr H)_1 = \left\{ \begin{bmatrix} e & 0 & \cdots & 0 \\ 0 & g_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & g_n \end{bmatrix} \;\middle|\; g_2, \ldots, g_n \in G \right\}$$

the *stabilizer of 1* in $G \wr H$.[1] Clearly, $|(G \wr H)_1| = |G|^{n-1}$. Now, define the subgroup

$$R = \left\{ h \begin{bmatrix} g & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g \end{bmatrix} \;\middle|\; h \in H;\; g \in G \right\}.$$

[1]The 1-stabilizer of a permuatation representation is usually defined to contain any element with 1 in the upper-lefthand entry.

It is clear that, $|R| = |H||G|$. With some inspection we see that R is a set of coset representatives for $(G \wr H)_1$. As we will see, structuring $G \wr H$ in this way is very useful.

Example 21. Let $G = \{e, g\}$, and let $H = C_3 = \left\{ \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} \right\}$. Then

$$(G \wr H)_1 = \left\{ \begin{bmatrix} e&0&0\\0&e&0\\0&0&e \end{bmatrix}, \begin{bmatrix} e&0&0\\0&e&0\\0&0&g \end{bmatrix}, \begin{bmatrix} e&0&0\\0&g&0\\0&0&e \end{bmatrix}, \begin{bmatrix} e&0&0\\0&g&0\\0&0&g \end{bmatrix} \right\},$$

and

$$R = \left\{ \begin{bmatrix} e&0&0\\0&e&0\\0&0&e \end{bmatrix}, \begin{bmatrix} g&0&0\\0&g&0\\0&0&g \end{bmatrix}, \begin{bmatrix} 0&0&e\\e&0&0\\0&e&0 \end{bmatrix}, \begin{bmatrix} 0&0&g\\g&0&0\\0&g&0 \end{bmatrix}, \begin{bmatrix} 0&e&0\\0&0&e\\e&0&0 \end{bmatrix}, \begin{bmatrix} 0&g&0\\0&0&g\\g&0&0 \end{bmatrix} \right\}.$$

Thus, $G \wr H = \{e, g\} \wr C_3 = \{rg_1 \mid r \in R\,;\ g_1 \in (G \wr H)_1\}$. △

For each $h \in H$, define

$$R_h = \left\{ h \begin{bmatrix} g & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & g \end{bmatrix} \;\middle|\; g \in G \right\}.$$

Example 22. Let G and H be defined as in example 21. Let $h_0 = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$, $h_1 = \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}$, and $h_2 = \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix}$. Then $G \wr H =$

$$\begin{array}{ll}
R_{h_0}(G \wr H)_1 & \left\{ \begin{array}{llll} \begin{bmatrix} e&0&0\\0&e&0\\0&0&e \end{bmatrix} & \begin{bmatrix} e&0&0\\0&e&0\\0&0&g \end{bmatrix} & \begin{bmatrix} e&0&0\\0&g&0\\0&0&e \end{bmatrix} & \begin{bmatrix} e&0&0\\0&g&0\\0&0&g \end{bmatrix} \Big\} (G \wr H)_1 \\ \begin{bmatrix} g&0&0\\0&g&0\\0&0&g \end{bmatrix} & \begin{bmatrix} g&0&0\\0&g&0\\0&0&e \end{bmatrix} & \begin{bmatrix} g&0&0\\0&e&0\\0&0&g \end{bmatrix} & \begin{bmatrix} g&0&0\\0&e&0\\0&0&e \end{bmatrix} \end{array} \right. \\
R_{h_1}(G \wr H)_1 & \left\{ \begin{array}{llll} \begin{bmatrix} 0&0&e\\e&0&0\\0&e&0 \end{bmatrix} & \begin{bmatrix} 0&0&e\\e&0&0\\0&g&0 \end{bmatrix} & \begin{bmatrix} 0&0&e\\g&0&0\\0&e&0 \end{bmatrix} & \begin{bmatrix} 0&0&e\\g&0&0\\0&g&0 \end{bmatrix} \\ \begin{bmatrix} 0&0&g\\g&0&0\\0&g&0 \end{bmatrix} & \begin{bmatrix} 0&0&g\\g&0&0\\0&e&0 \end{bmatrix} & \begin{bmatrix} 0&0&g\\e&0&0\\0&g&0 \end{bmatrix} & \begin{bmatrix} 0&0&g\\e&0&0\\0&e&0 \end{bmatrix} \end{array} \right. \\
R_{h_2}(G \wr H)_1 & \left\{ \begin{array}{llll} \begin{bmatrix} 0&e&0\\0&0&e\\e&0&0 \end{bmatrix} & \begin{bmatrix} 0&e&0\\0&0&e\\g&0&0 \end{bmatrix} & \begin{bmatrix} 0&e&0\\0&0&g\\e&0&0 \end{bmatrix} & \begin{bmatrix} 0&e&0\\0&0&g\\g&0&0 \end{bmatrix} \\ \begin{bmatrix} 0&g&0\\0&0&g\\g&0&0 \end{bmatrix} & \begin{bmatrix} 0&g&0\\0&0&g\\e&0&0 \end{bmatrix} & \begin{bmatrix} 0&g&0\\0&0&e\\g&0&0 \end{bmatrix} & \begin{bmatrix} 0&g&0\\0&0&e\\e&0&0 \end{bmatrix} \end{array} \right.
\end{array}.$$

Note that the leftmost column is simply R, while the top row is $(G \wr H)_1$. △

This partitioning of $G \wr H$ into $\bigcup_{h \in H} R_h(G \wr H)_1$ will prove very useful in characterizing its polytopes in section 3.2.

3.2 Polytopes of Wreath Products

Proposition 23. *Given a representation of a finite group* G*, its associated polytope* $P(G)$*, and a permutation group* $H \subseteq S_n$*, let* R_h *and* $(G \wr H)_1$ *be defined as in section 3.1. Then*

$$P(R_h(G \wr H)_1) \cong P(G)^n$$

for each h *in* H*, where the exponent refers to cartesian product. Furthermore, if* H *is regular, and if* h *and* h' *are distinct elements of* H*, then* $P(R_h(G \wr H)_1)$ *and* $P(R_{h'}(G \wr H)_1)$ *lie in complementary spaces.*

Proof. Notice that each $P(\mathbb{R}_h(G \wr H)_1)$ is just a change of coordinates of the polytope $P(\mathbb{R}_e(G \wr H)_1)$, so it will suffice to show that the first part of theorem holds for

$$P(R_e(G \wr H)_1) = \text{conv}\left(\left\{ \begin{bmatrix} g_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_n \end{bmatrix} \;\middle|\; g_i \in G \right\}\right).$$

By rearranging the coordinates to put all the zeros at the end, we get

$$P(R_e(G \wr H)_1) = \text{conv}\left(\{(g_1, \ldots, g_n, \vec{0}\}\right),$$

which is precisely the definition of $P(G)^n$ embedded in a larger space.

To see that the $P(G)^n$ associated with distinct elements of a regular group H sit in complementary spaces, one need only notice that the linear spans of the polytopes intersect only at the origin. □

We thus have $P(G \wr H)$ as the convex hull of $|H|$ complementary polytopes. Theorem 24 describes how these subpolytopes produce the polytope of the wreath product.

Theorem 24. *Given a representation of a finite group* G*, its corresponding polytope* $P(G)$*, and a regular permutation group* $H \subseteq S_n$*,*

$$P(G \wr H) \cong \begin{cases} [P(G)^n]^{\oplus |H|} & \textit{if the origin is in } \text{relint}(P(G)) \\ [P(G)^n]^{\bowtie |H|} & \textit{otherwise.} \end{cases}$$

Proof. The proof of this theorem will require two cases.

<u>Case 1</u>: $\vec{0} \in \text{relint}(P(G))$

Since H is regular, the affine spans of each $P(R_h(G \wr H)_1)$ intersect either at the origin or nowhere. But because the origin is in the relative interior of $P(G)$, it must be in the relative interior of $P(G)^n$, and the polytopes $P(R_h(G \wr H)_1)$ intersect only at the origin, which is in the relative interior of each. Thus, $P(G \wr H) \cong [P(G)^n]^{\oplus |H|}$.

Case 2: $\vec{0} \notin \text{relint}(P(G))$

We want to show that in this case the affine spans of two polytopes $P(R_h(G \wr H)_1)$ and $P(R_{h'}(G \wr H)_1)\}$ must be skew. That is, we want to show that the affine spans of the spaces contain no parallel lines and do not intersect. We know by proposition 23 that the spaces are complementary, and so contain no parallel lines. Then, because the only possible point of intersection is the origin, we need only show that neither has the origin in its affine span.

To do this, assume that the origin $\vec{0} \in \text{aff}(P(G))$. Now, let $x = \frac{1}{|G|}\sum_{g \in G} g$. If we multiply both sides of this equation on the left by some particular $g_0 \in G$, we have

$$g_0 x = \frac{1}{|G|} g_0 \sum_{g \in G} g = \frac{1}{|G|} \sum_{g \in G} g_0 g = x.$$

Thus, $gx = x$ for all g, i.e., x is invariant under G. Now, because $\vec{0} \in \text{aff}(P(G))$, we have $\sum \lambda_g g = \vec{0}$ for some $\{\lambda_g \mid g \in G\ ;\ \sum \lambda_g = 1\}$. Therefore, $(\sum \lambda_g g)\, x = \vec{0}$, and

$$\sum \lambda_g g x = \sum \lambda_g x = \left(\sum \lambda_g\right) x = \vec{0}.$$

Because $\sum \lambda_g = 1$, this means that $x = \vec{0}$. Also, notice that $x = \sum \frac{1}{|G|} g$ is in $\text{relint}(P(G))$, contradicting our condition that $\vec{0} \notin \text{relint}(P(G))$ and showing that $\vec{0} \notin \text{aff}(P(G))$. Thus, given distinct elements $h, h' \in H$, $\text{aff}(P(R_h(G \wr H)_1))$ and $\text{aff}(P(R_{h'}(G \wr H)_1))$ must be skew. Therefore, taking the convex hull of the polytopes in the set $\{P(R_h(G \wr H)_1) \mid h \in H\}$ gives the free join. That is, $P(G \wr H) \cong [P(G)^n]^{\bowtie |H|}$. ☐

Corollary 25. *If* $P(G)$ *is a* d*-polytope and* H *is a regular subgroup of* S_n*, then*

$$\dim\left(P(G \wr H)\right) = \begin{cases} nd|H| & \text{if } \vec{0} \in \text{relint}(P(G)) \\ nd|H| + |H| - 1 & \text{if } \vec{0} \notin \text{relint}(P(G)). \end{cases}$$

Proof. This is clear from the dimensions of cartesian products, direct sums, and free joins. ☐

Remark 26. The combinatorial structure of $P(G \wr H)$ — that is, the structure of the face lattice $\mathcal{L}(P(G \wr H))$ — now follows directly from theorems 24 and 10.

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