The CW Decomposition of the Grassmannian Manifold

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Abstract

This paper presents the standard CW decomposition of the real Grassmannian manifold. We begin by giving a general description of both the Grassmannian and CW decompositions. We conclude with a proof that the decomposition presented is indeed a CW decomposition.

Introduction

The Grassmannian $\mathrm{Gr}(k, n)$ parameterizes the set of k-dimensional subspaces of an n-dimensional vector space. It is a key ingredient for the foundation of Schubert calculus, which was created by Hermann Schubert in the nineteenth century. Schubert was interested in solving various counting problems of enumerative geometry. For example, a typical problem might be "how many lines in 3-space, in general, intersect four given lines?" Grassmannians are also useful in mathematics in the creation of parameter spaces for objects that are more complicated than linear spaces.

One component of the analysis of the Grassmannian is its CW decomposition, i.e., a method of constructing $\mathrm{Gr}(k, n)$ by gluing together open balls of various dimensions. The collection of cells of a CW decomposition, which are homeomorphic to these open balls, form a CW complex. More generally, a CW complex is a type of topological space that was first introduced by J.H.C. Whitehead in the mid-twentieth century and remains important in algebraic topology. The purpose of this thesis is to provide an exposition of the CW decomposition of $\mathrm{Gr}(k, n)$ suitable for advanced undergraduates. CW decompositions of $\mathrm{Gr}(k, n)$ have been discussed and proven by various mathematicians. Our main reference is the book *Characteristic Classes* by Milnor & Stasheff (1974). We were also helped by notes from a course by Mark Hopkins (2014).

Chapter 1 outlines the general definition of $\mathrm{Gr}(k, n)$ and puts a manifold structure on $\mathrm{Gr}(k, n)$. Examples of $\mathrm{Gr}(k, n)$ with matrix representatives are also provided. The chapter ends with a brief discussion of Plücker coordinates. Chapter 2.1 provides general definitions and examples relevant to CW complexes and CW decompositions. More broadly, Chapter 2.2 constructs the CW decomposition of $\mathrm{Gr}(k, n)$. It starts by discussing the Schubert cells of the Grassmannian, before briefly discussing Young diagrams in Section 2.2.1. Section 2.2.2 defines Schubert varieties. Lastly, Section 2.2.3 gives the characteristic maps necessary to prove that the Schubert cells do indeed form a CW decomposition of $\mathrm{Gr}(k, n)$. An example of why $\mathrm{Gr}(k, n)$ is not a *regular* CW complex is also provided.

Chapter 1

Manifold structure of the Grassmannian

Fix non-negative integers $k \leq n$. Define $\mathrm{Gr}(k,n)$ as the set of k-dimensional subspaces of $\mathbb{R}^n$. Our first goal is to give "coordinates" of points in $\mathrm{Gr}(k,n)$. Let $M_{k\times n}$ denote the real matrices with k rows and n columns. Given a k dimensional subspace $V \subseteq \mathbb{R}^n$, choose a basis $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ for V, and let $L \in M_{k\times n}$ be the $k \times n$ matrix whose rows are $v_1, \ldots, v_k$. We can then use L to represent V, since L determines the subspace $V \subseteq \mathbb{R}^n$ by its rowspace. Every $k \times n$ matrix of rank k determines an element of $\mathrm{Gr}(k,n)$. Note that there are multiple options for the basis of V, so L depends on the choice of $v_1, v_2, \ldots, v_k$. This results in two matrices representing the same V differing by elementary row operations.

We would like to choose a favorite representative in each equivalence class so that the set of $L \in \mathrm{Gr}(k,n)$ is in bijection with the set of k dimensional subspaces of $\mathbb{R}^n$. First, let $M^*_{k\times n} \subseteq M_{k\times n}$ denote the $k \times n$ matrices of full rank k (determining rank will be discussed shortly). We will define an equivalence class on $M^*_{k\times n}$ as $L \sim L'$ if there exists some invertible $k \times k$ matrix A such that $L' = AL$ (which is equivalent to L and L' differing by row operations).

Example 1. Let $V = \mathrm{Span}\{(1,3,3,0),(0,1,0,1)\}$. One matrix L that represents V is

$$L = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Consider L and L', where L' is L with elementary row operations applied.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 6 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \implies \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & 6 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}.$$

We see that the k-dimensional $V \subseteq \mathbb{R}^n$ are in bijection with the equivalence classes for $\sim$, since two matrices representing V that differ by elementary row operations are in the same equivalence class. Thus, we now redefine $\mathrm{Gr}(k,n)$ to be $M^*_{k\times n}/\sim$, and we have the projection map

$$\begin{aligned} \pi : M^*_{k\times n} &\to \mathrm{Gr}(k,n) \\ L &\mapsto [L] \end{aligned} \tag{1.1}$$

with $[L]$ as the equivalence class of L defined by $\sim$.

Topology. Our next goal is to put a topology on $\mathrm{Gr}(k,n)$. First, consider the $k \times k$ submatrices of some $L \in M_{k\times n}$. Let $\binom{[n]}{k}$ be the collection of k element subsets of $[n] := \{1, 2, ..., n\}$. Given $L \in M_{k\times n}$ and $\Lambda \in \binom{[n]}{k}$, let L_Λ be the $k \times k$ submatrix of L consisting of the columns indexed by Λ in increasing order. Further, the *minor* of L corresponding to Λ is $\Delta_\Lambda(L) := \det(L_\Lambda)$. For example, let

$$L = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

If $\Lambda = \{1, 3\} \in \binom{[4]}{2}$, we obtain the 2×2 submatrix

$$L_\Lambda = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta_\Lambda(L) = 0.$$

By elementary linear algebra, we have the following characterization of rank.

Proposition 2. *Let $L \in M_{k\times n}$. Then the rank of L is k if and only if $\Delta_\Lambda(L) \neq 0$ for some $\Lambda \in \binom{[n]}{k}$.*

Now, identify $M_{k\times n}$ with $\mathbb{R}^{kn}$ by reading off the elements of $M_{k\times n}$ left-to-right and top-to-bottom. Then given $M_{k\times n}$, the topology it inherits is the ordinary topology of $\mathbb{R}^{kn}$.

Proposition 3. *$M^*_{k\times n}$ is an open subset of $M_{k\times n}$.*

Proof. First, we will define the complement $X \subseteq M_{k\times n}$ of $M^*_{k\times n}$:

$$X = \{L \in M_{k\times n} : \mathrm{rank}(L) < k\} = \left\{L \in M_{k\times n} : \Delta_\Lambda(L) = 0 \ \forall \Lambda \in \binom{[n]}{k}\right\}.$$

To prove the proposition, we need to show that X is closed. Denoting the determinant mapping $\Delta : M_{k\times k} \to \mathbb{R}$, recall that

$$X = \bigcap_{\Lambda \in \binom{[n]}{k}} \{L \in M_{k\times n} : \Delta_\Lambda(L) = 0\} = \bigcap_{\Lambda \in \binom{[n]}{k}} \Delta_\Lambda^{-1}(0).$$

Since Δ is a polynomial function of the entries of its argument, it is continuous. Then, since $\{0\} \subset \mathbb{R}$ is closed, $\Delta^{-1}\{0\}$ is closed. So X is a finite intersection of closed sets, and hence is closed. $\square$

Define the topology on $\mathrm{Gr}(k,n)$ to be the quotient topology via the projection map (1.1). Thus, $U \subseteq \mathrm{Gr}(k,n)$ is open if and only if $\pi^{-1}(U)$ is open in $M^*_{k\times n}$.

Manifold Structure. Now that we have defined the topology on the Grassmannian, we can define the manifold structure. For $\Lambda \in \binom{[n]}{k}$, define

$$U_\Lambda := \{[L] \in \mathrm{Gr}(k,n) : \Delta_\Lambda(L) \neq 0\}.$$

Note that U_Λ is well-defined since performing a row operation on L will not change the fact that $\Delta_\Lambda(L) \neq 0$. In detail, suppose $L = AL'$ for some $A \in M^*_{k\times k}$. Then

$$\begin{aligned}\Delta_\Lambda(L') &= \Delta_\Lambda(AL) = \det((AL)_\Lambda)\\ &= \det(AL_\Lambda) = \det(A)\det(L_\Lambda) = \det(A)\Delta_\Lambda(L).\end{aligned}$$

Since $\det(A) \neq 0$, we have $\Delta_\Lambda(L') \neq 0$ if and only if $\Delta_\Lambda(L) \neq 0$.

Given $\Lambda \in \binom{[n]}{k}$, define the homeomorphism

$$\phi_\Lambda : U_\Lambda \to M_{k\times(n-k)} \cong \mathbb{R}^{k(n-k)}$$

as follows: given $[L] \in U_\Lambda$,

1. Apply row operations to L to obtain the matrix K such that $K_\Lambda = I_{k\times k}$, the $k \times k$ identity matrix.
2. Remove the columns indexed by Λ in K to create $K^\Lambda \in M_{k\times(n-k)}$.
3. Define $\phi_\Lambda([L]) = K^\Lambda$.

See Example 4 below. Note that ϕ_Λ is well-defined since the reduced row-echelon form of a matrix is unique. Additionally, $U_\Lambda \subseteq \mathrm{Gr}(k, n)$ is an open since the set $[L]$ such that all $\Delta_\Lambda = 0$ is closed. Thus, U_Λ maps homeomorphically to $\phi_\Lambda(U_\Lambda) = M_{k\times(n-k)} \cong \mathbb{R}^{k(n-k)}$. The manifold structure on $\mathrm{Gr}(k, n)$ is given by the atlas

$$\{(U_\Lambda, \phi_\Lambda)\}_{\Lambda\in\binom{[n]}{k}}.$$

Then $\mathrm{Gr}(k, n)$ is locally Euclidean since every point of $\mathrm{Gr}(k, n)$ belongs to some chart domain of $\mathrm{Gr}(k, n)$, i.e., $\mathrm{Gr}(k, n) = \bigcup_{\Lambda\in\binom{[n]}{k}} U_\Lambda$.

Example 4. Using L from Example 1, let $\Lambda_1 = (1, 2)$ and $\Lambda_2 = (3, 4)$. Applying steps given above, we obtain

$$\Lambda_1 : \begin{bmatrix}1 & 3 & 3 & 0\\ 0 & 1 & 0 & 1\end{bmatrix} \text{ reduces to } \begin{bmatrix}1 & 0 & 3 & -3\\ 0 & 1 & 0 & 1\end{bmatrix}, \text{ so } K^{\Lambda_1} = \begin{bmatrix}3 & -3\\ 0 & 1\end{bmatrix},$$

$$\Lambda_2 : \begin{bmatrix}1 & 3 & 3 & 0\\ 0 & 1 & 0 & 1\end{bmatrix} \text{ reduces to } \begin{bmatrix}1/3 & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}, \text{ so } K^{\Lambda_2} = \begin{bmatrix}1/3 & 1\\ 0 & 1\end{bmatrix}.$$

Using $\{U_{\Lambda_1}, \phi_{\Lambda_1}\}$ and $\{U_{\Lambda_2}, \phi_{\Lambda_2}\}$, with the values as assigned above, we can define the transition map from ϕ_{Λ_1} to ϕ_{Λ_2} as

$$(\phi_{\Lambda_2} \circ \phi^{-1}_{\Lambda_1}) = \phi_{\Lambda_1,\Lambda_2}.$$

Defining the matrices as coordinates in $\mathbb{R}^4$ we obtain

$$\phi_{\Lambda_1,\Lambda_2}(3, -3, 0, 1) = (1/3, 1, 0, 1).$$

Now we seek to define these *transition functions* more generally. The transition function from U_{Λ_i} to U_{Λ_j} is defined as

$$\phi_{\Lambda_i,\Lambda_j} := \phi_{\Lambda_j} \circ \phi_{\Lambda_i}^{-1}.$$

These transition functions are defined on a domain of $U_{\Lambda_i} \cap U_{\Lambda_j}$, which is the collection of matrices in $\mathrm{Gr}(k,n)$ such that both $\Delta_{\Lambda_i}, \Delta_{\Lambda_j} \neq 0$.

Example 5. To further elaborate, we will define a new matrix $L \in \mathrm{Gr}(k,n)$ and compute the transition function $\phi_{\Lambda_1,\Lambda_2}$ with $\Lambda_1 = (1,2), \Lambda_2 = (2,4)$. Let

$$L = \begin{bmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{bmatrix}, \text{ with } a,b,c,d,e,f \in \mathbb{R} \text{ such that } b \neq 0.$$

This gives

$$\phi_{\Lambda_1}([L]) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Taking columns 2 and 4 of L, we can define the matrix A with its inverse:

$$A = \begin{bmatrix} 0 & b \\ 1 & e \end{bmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} e & -b \\ -1 & 0 \end{bmatrix} = \frac{1}{-b} \begin{bmatrix} e & -b \\ -1 & 0 \end{bmatrix}.$$

Due to how A has been defined, $A^{-1}L$ will have columns 2 and 4 as the identity matrix. After columns 2 and 4 have been turned into the identity matrix, $\phi_{\Lambda_2}(L)$ omits those two columns, so they can be omitted before multiplication as follows:

$$\phi_{\Lambda_2}(L) = (L_{\Lambda_2})^{-1}(L_{\Lambda_2}) = A^{-1} \begin{bmatrix} 1 & a & c \\ 0 & d & f \end{bmatrix} = \begin{bmatrix} -\frac{e}{b} & -\frac{ea-bd}{b} & -\frac{ec-bf}{b} \\ \frac{1}{b} & \frac{a}{b} & \frac{c}{b} \end{bmatrix}$$
$$\phi_{\Lambda_1,\Lambda_2}\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \right) = \begin{bmatrix} -\frac{e}{b} & -\frac{ea-bd}{b} & -\frac{ec-bf}{b} \\ \frac{1}{b} & \frac{a}{b} & \frac{c}{b} \end{bmatrix}.$$

Notice that each entry of the matrix resulting from $\phi_{\Lambda_1,\Lambda_2}$ (and $\phi_{\Lambda_i,\Lambda_j}$ more generally) is a rational function of the entries of its argument. Additionally, since $\Delta_{\Lambda_i}, \Delta_{\Lambda_j} \neq 0$, there is no division by zero, so these transition functions are both rational and well-defined, and thus, differentiable.

Plücker Coordinates. The *Plücker coordinates* for $[L] \in \mathrm{Gr}(k,n)$ are the minors $\Delta_\Lambda(L)$ for L listed lexicographically according to Λ:

$$(\Delta_{\{1,2,\ldots,k\}}(L), \Delta_{\{1,2,\ldots,k+1\}}(L), \ldots, \Delta_{\{2,3,\ldots,k+1\}}(L), \ldots, \Delta_{\{n-k,n-k+1,\ldots,n\}}(L)).$$

For instance, if L is the specific 2×4 matrix given in Example 4, then the Plücker coordinates for $[L] \in \mathrm{Gr}(k,n)$ are

$$(\Delta_{\{1,2\}}(L), \Delta_{\{1,3\}}(L), \Delta_{\{1,4\}}(L), \Delta_{\{2,3\}}(L), \Delta_{\{2,4\}}(L), \Delta_{\{3,4\}}(L)) = (1,0,1,-3,2,3).$$

By Proposition 6, the Plücker coordinates are well defined up to a non-zero scalar.

Proposition 6. *The Plücker coordinates of* $[L] \in \mathrm{Gr}(k,n)$ *are well-defined as a point in projective space. Thus, we have a mapping, called the* Plücker embedding, *defined by*

$$\begin{aligned}\Gamma\colon\ \mathrm{Gr}(k,n) &\to \mathbb{P}^{\binom{n}{k}-1}\\ [L] &\mapsto (\Delta_\Lambda([L]))_\Lambda.\end{aligned}$$

Proof. By Proposition 2, any representative of $[L]$ will have some $\Delta_\Lambda(L) \neq 0$. Let L and L' both be representatives of $[L]$. Because L and L' differ by row operations, there exists some matrix $A \in M^*_{k\times k}$ such that $L = AL'$. Thus, $L_\Lambda = AL'_\Lambda$. This implies that $\Delta_\Lambda(L) = \det(A)\Delta_\Lambda(L')$ for all $\Lambda \in \binom{[n]}{k}$. This gives $(\Delta_\Lambda(L))_\Lambda = \gamma(\Delta_\Lambda(L'))_\Lambda$ with $\gamma = \det(A) \neq 0$. Since the Plücker coordinates of L and L' are scalar multiples of one another, they both map to the same point in projective space. Hence, the Plücker embedding maps all $[L] \in \mathrm{Gr}(k,n)$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}$. □

Proposition 7. *The Plücker embedding is a smooth embedding of manifolds, and its image is given by the* Grassmann-Plücker relations*:*

$$\Delta_{(i_1,\ldots,i_k)} \cdot \Delta_{(j_1,\ldots,j_k)} = \sum_{s=1}^{k} \Delta_{(s,i_2,\ldots,i_k)} \cdot \Delta_{(j_1,\ldots,j_{s-1},i_1,j_{s+1},\ldots,j_k)}$$

for any $i_1,\ldots,i_k,j_1,\ldots,j_k \in [n]$, *with* $(i_1,\ldots,i_k)$ *as an ordered sequence instead of a subset.*

Proof. See (Kleiman & Laksov, 1972, Theorem 1). □

Chapter 2

CW decomposition of $\mathrm{Gr}(k, n)$

2.1 CW complexes

We seek to define a CW complex more generally, and then define the CW decomposition of $\mathrm{Gr}(k, n)$. First, define the *n-dimensional disk* (or n-disk) in $\mathbb{R}^n$ as

$$D^n = \{x \in \mathbb{R}^n : |x| \leq 1\},$$

which is a closed subset of $\mathbb{R}^n$. The *open n-disk* in $\mathbb{R}^n$ is the interior of D^n,

$$B^n = \mathrm{int}(D^n) = \{x \in \mathbb{R}^n : |x| < 1\}.$$

The *boundary* of $D^n \subset \mathbb{R}^n$ is the $(n-1)$-sphere,

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

For example, since $\mathbb{R}^0 = \{0\}$, by definition, we have

$$D^0 = \{0\} = B^0.$$

The 1-disk is $D^1 = [-1, 1]$, with $B^1 = (-1, 1)$, and boundary $S^0 = \{-1, 1\}$.

Definition 8. An *n-cell* is a space homeomorphic to B^n. A *cell*, more generally, is an n-cell for some $n \geq 0$.

Definition 9. A *cell decomposition* of a space X is a collection $\mathcal{E} = \{e(\alpha) : \alpha \in I\}$ (where I is an indexing set) of subspaces of X such that each $e(\alpha)$ is a cell and X is the disjoint union

$$X = \bigsqcup_{\alpha \in I} e(\alpha).$$

The *n-skeleton* of the cell decomposition, denoted X^n, is

$$X^n = \bigsqcup_{\alpha \in I : \dim(e(\alpha)) \leq n} e(\alpha).$$

A *finite cell decomposition* is a cell decomposition consisting of finitely many cells, i.e. $|I| < \infty$.

Definition 10. A *CW complex* is a pair $(X, \mathcal{E})$ consisting of a Hausdorff topological space X with cell decomposition $\mathcal{E}$ of X with the following properties:

1. (Characteristic Maps) For each n-cell $e \in \mathcal{E}$, there exists a map $\Phi_e\colon D^n \to X$. Furthermore, (i) restricting Φ_e to the interior of D^n gives a homeomorphism

$$\Phi_e|_{B^n}\colon B^n \to e,$$

and (ii) the image of Φ_e restricted to the boundary of D^n is contained in the $(n-1)$-skeleton of the CW complex,

$$\Phi_e(S^{n-1}) \subseteq X^{n-1}.$$

2. (Closure Finiteness) For any cell $e \in \mathcal{E}$, the closure $\overline{e}$ intersects only a finite number of other $e \in \mathcal{E}$.

3. (Weak Topology) A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e}$ is closed in X for all $e \in \mathcal{E}$.

Reflecting axioms 2 and 3, the C in CW stands for "closure-finite," and the W stands for "weak" topology. If $(X, \mathcal{E})$ is a CW complex, then $\mathcal{E}$ is a *CW decomposition* of X. A *finite CW complex* is a CW complex with a finite cell decomposition (see Example 13).

The following proposition shows that to define a finite CW complex, Properties 2 and 3 are unnecessary.

Proposition 11. *Let $\mathcal{E}$ be a finite cell decomposition of a space X. Then, Properties 2 and 3 of Definition 10 hold.*

Proof. Axiom 2 is holds since $\mathcal{E}$ is finite. To prove Property 3, first suppose $A \subseteq X$ is closed. Then $A \cap \overline{e}$ is the intersection of two closed set and, hence, is closed. Conversely, now suppose $A \cap \overline{e}$ is closed for all $e \in \mathcal{E}$. Then,

$$A = A \cap X = A \cap \bigcup_{e \in \mathcal{E}} \overline{e} = \bigcup_{e \in \mathcal{E}} (A \cap \overline{e}).$$

We have written A as a finite union of closed sets. So A is closed. $\square$

Proposition 12. *Let $(X, \mathcal{E})$ be a Hausdorff space X with a cell decomposition $\mathcal{E}$.*

(i) If $(X, \mathcal{E})$ satisfies Axiom 1 from Definition 10, then we have

$$\overline{e} = \Phi_e(D^n)$$

for all $e \in \mathcal{E}$.

(ii) Furthermore, $\overline{e}$ is a compact subspace of X and the cell boundary $\overline{e} \backslash e = \Phi_e(S^{n-1})$ lies in X^{n-1}.

Proof. (i) Since Φ_e is continuous, $\Phi_e(\overline{B^n}) \subseteq \overline{\Phi_e(B^n)}$. Because $\Phi_e(B^n) = e$ and $B^n \subseteq D^n = \overline{B^n}$, we obtain

$$e = \Phi_e(B^n) \subseteq \Phi_e(D^n) = \Phi_e(\overline{B^n}) \subseteq \overline{\Phi_e(B^n)} = \bar{e}.$$

Note, since D^n is compact and Φ_e is continuous, $\Phi_e(D^n)$ is also compact. Then, since X is Hausdorff, $\Phi_e(D^n)$ is closed. Since $\Phi_e(D^n)$ is closed and contains e, it follows that $\bar{e} \subseteq \Phi_e(D^n)$. We have now shown that $\bar{e} \subseteq \Phi_e(D^n)$ and $\bar{e} \supseteq \Phi_e(D^n)$, thus, $\bar{e} = \Phi_e(D^n)$.

(ii) First we will prove that $\Phi_e(S^{n-1}) \supseteq \bar{e} \setminus e$. Let $y \in \bar{e} \setminus e$. Then, $y \in \Phi_e(D^n) = \bar{e}$. There exists $x \in D^n$ such that $\Phi_e(x) = y$. If $x \in B^n$, then $\Phi_e(x) = y \in e$. However, we assumed $y \notin e$. Therefore, $x \in D^n \setminus B^n = S^{n-1}$.

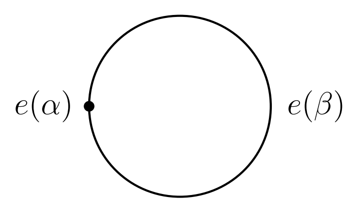
Now we will prove the opposite inclusion. Since $\dim(e) = n$, we have $X^{n-1} \cap e = \emptyset$, which implies that $\Phi_e(S^{n-1}) \cap e = \emptyset$. Furthermore, $\Phi_e(S^{n-1}) \subseteq \Phi_e(D^n) = \bar{e}$. Thus, $\Phi_e(S^{n-1}) \subseteq \bar{e} \setminus e$. Since we have show both directions of containment, we obtain the desired $\Phi_e(S^{n-1}) = \bar{e} \setminus e$. □

Example 13. We give two examples of CW decompositions of S^1:

1. Let $\mathcal{E} = \{e(\alpha), e(\beta)\}$ where $e(\alpha) = \{(-1,0)\}$ and $e(\beta) = S^1 \setminus \{(-1,0)\}$, with characteristic maps

$$\begin{aligned} \Phi_{e(\alpha)} : B^0 &\to S^1 \\ 0 &\mapsto (-1,0) \end{aligned} \qquad \begin{aligned} \Phi_{e(\beta)} : B^1 &\to S^1 \\ t &\mapsto (\cos \pi t, \sin \pi t). \end{aligned}$$

Note that $B^1 = (-1,1)$, so for $\Phi_{e(\beta)}$, t goes over the interval $(-1,1)$. Here is an illustration of the decomposition:



2. Let $\mathcal{E}' = \{e(\alpha)', e(\beta)', e(\gamma)', e(\delta)'\}$ where

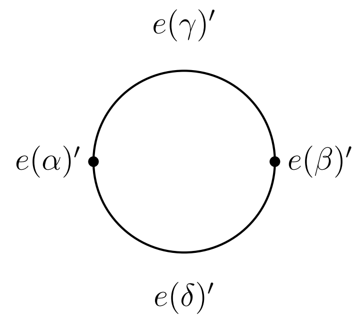
$$\begin{aligned} e(\alpha)' &= \{(-1,0)\} \\ e(\beta)' &= \{(1,0)\} \end{aligned} \qquad \begin{aligned} e(\gamma)' &= \{(x,y) \in S^1 : y > 0\} \\ e(\delta)' &= \{(x,y) \in S^1 : y < 0\}, \end{aligned}$$

with the characteristic maps

$$\begin{aligned} \Phi_{e(\alpha)'} \colon B^0 &\to S^1 \\ 0 &\mapsto (-1,0) \end{aligned} \qquad \begin{aligned} \Phi_{e(\gamma)'} \colon B^1 &\to S^1 \\ t &\mapsto (t, \sqrt{1-t^2}) \end{aligned}$$

$$\begin{aligned} \Phi_{e(\beta)'} \colon B^0 &\to S^1 \\ 0 &\mapsto (1,0) \end{aligned} \qquad \begin{aligned} \Phi_{e(\delta)'} \colon B^1 &\to S^1 \\ t &\mapsto (t, -\sqrt{1-t^2}). \end{aligned}$$

Below is an illustration of this decomposition:



Definition 14. A CW complex is a *regular CW complex* if for each $e \in \mathcal{E}$, the continuous map Φ_e from the n-dimensional closed ball to X is a homeomorphism onto $\bar{e}$.

For example, Part 1 from Example 13 is not regular. The closure of B^1 is the closed interval $D^1 = [-1, 1]$. For $t = -1$ and $t = 1$, we obtain

$$\Phi_{e(\beta)}(-1) = (-1, 0) = \Phi_{e(\beta)}(1).$$

Thus, $\Phi_{e(\beta)}$ is not injective, hence, not a homeomorphism onto its image. However, part 2 of Example 13 is indeed regular. The maps $\Phi_{e(\alpha)'}$ and $\Phi_{e(\beta)'}$ with the domain $\{0\}$ are clearly homeomorphisms. To show that $\Phi_{e(\gamma)'}$ and $\Phi_{e(\delta)'}$ are also homeomorphisms, consider their inverses:

$$(\Phi_{e(\gamma)'})^{-1} \colon \overline{e(\gamma)'} \to D^1 \qquad\qquad (\Phi_{e(\delta)'})^{-1} \colon \overline{e(\delta)'} \to D^1$$
$$(x, y) \mapsto x \qquad\qquad (x, y) \mapsto x.$$

These inverses are clearly well-defined on both $\overline{e(\gamma)'} = \{(x, y) \in S^1 : y \geq 0\}$ and $\overline{e(\delta)'} = \{(x, y) \in S^1 : y \leq 0\}$. Both functions have D^1 as their images.

2.2 Schubert cell decomposition of Gr(k, n)

First, define the *flag* $\mathcal{R}$ of linear subspaces of $\mathbb{R}^n$ as

$$\mathcal{R} = \mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^n.$$

Here, for each $i \in [n]$, we consider $\mathbb{R}^i$ as a subspace of $\mathbb{R}^n$ via

$$\mathbb{R}^i \cong \{(x_1, \ldots, x_i, 0, \ldots, 0) : x_j \in \mathbb{R}\} \subset \mathbb{R}^n.$$

Let $X \in \mathbb{R}^n$ be a k-dimensional subspace. This produces a sequence of integers

$$0 \leq \dim(X \cap \mathbb{R}^1) \leq \dim(X \cap \mathbb{R}^2) \leq \cdots \leq \dim(X \cap \mathbb{R}^n) = k.$$

Proposition 15. *No two consecutive integers in the above sequence will differ by more than one. More explicitly,*

$$\dim(X \cap \mathbb{R}^{i+1}) \leq \dim(X \cap \mathbb{R}^i) + 1.$$

Proof. Let $v_1, \ldots, v_\ell \in \mathbb{R}^i \subseteq \mathbb{R}^n$ be a basis for $X \cap \mathbb{R}^i$, and let's suppose there exists

$$v_{\ell+1} = (a_1, \ldots, a_{i+1}, 0, \ldots, 0) \in (X \cap \mathbb{R}^{i+1}) \setminus (X \cap \mathbb{R}^i).$$

In particular, $a_{i+1} \neq 0$ since $v_{\ell+1} \notin X \cap \mathbb{R}^i$. We claim that $\{v_1, \ldots, v_{\ell+1}\}$ is a basis for $X \cap \mathbb{R}^{i+1}$. First, note that $v_{\ell+1}$ is linearly independent from $\{v_1, \ldots, v_\ell\}$ since $v_{\ell+1} \notin \mathrm{Span}\{v_1, \ldots, v_\ell\} = X \cap \mathbb{R}^i$. Next, suppose $w = (b_1, \ldots, b_{i+1}, 0, \ldots, 0) \in X \cap \mathbb{R}^{i+1}$. Then, $w - \frac{b_{i+1}}{a_{i+1}} v_{\ell+1} \in X \cap \mathbb{R}^i$. So there exists some $c_1, \ldots, c_\ell \in \mathbb{R}$ such that $w - \frac{b_{i+1}}{a_{i+1}} v_{\ell+1} = c_1 v_1 + \cdots + c_\ell v_\ell$. But then,

$$w = c_1 v_1 + \cdots + c_\ell v_\ell + \frac{b_{i+1}}{a_{i+1}} v_{\ell+1} \in \mathrm{Span}\{v_1, \ldots, v_{\ell+1}\}.$$

Thus, $\{v_1, \ldots, v_{\ell+1}\}$ is indeed a basis for $X \cap \mathbb{R}^{i+1}$. Since the basis of $X \cap \mathbb{R}^{i+1}$ can have *at most* one more element than the basis of $X \cap \mathbb{R}^i$, we obtain $\dim(X \cap \mathbb{R}^{i+1}) \leq \dim(X \cap \mathbb{R}^i) + 1$. □

Definition 16. A *Schubert symbol* for Gr(k, n) is a sequence of integers $\sigma = (\sigma_1, \ldots, \sigma_k)$ satisfying

$$1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n.$$

The corresponding *Schubert cell* is defined as

$$e(\sigma) = \{X \in \mathrm{Gr}(k, n) : \dim(X \cap \mathbb{R}^{\sigma_i}) = \dim(X \cap \mathbb{R}^{\sigma_i - 1}) + 1 \text{ for } i = 1, \ldots, k\}.$$

Let $X \in e(\sigma)$ and suppose $X = [L]$ with $L \in M^*_{k \times n}$. Let $\tilde{L}$ be the matrix obtained from L by "upside down and backwards" row reduction. In other words, instead of reducing starting in the upper left, reduce starting in the lower right. Then, the basic columns of $\tilde{L}$ have indices $\sigma_1, \ldots, \sigma_k$. We will call $\tilde{L}$ the **canonical representative** of X.

Example 17. For example, in the case of Gr$(3, 9)$, the Schubert cell $e(2, 4, 7)$ consists of points $X = [L]$ with corresponding canonical representative $\tilde{L}$ of the form

$$\tilde{L} = \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & * & * & 1 & 0 & 0 \end{bmatrix},$$

where each $*$ is some element of $\mathbb{R}$.

Example 18. Consider the Schubert symbols $\sigma = (1, 2)$ and $\tau = (3, 4)$ for Gr$(2, 4)$. Then $e(\sigma)$ contains the single element represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and $e(\tau)$ contains elements $[L]$ where L has the form

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix},$$

for some $a, b, c, d \in \mathbb{R}$. Thus,

$$e(\sigma) = \{\mathrm{Span}\{(1,0,0,0),(0,1,0,0)\}\}$$
$$e(\tau) = \{\mathrm{Span}\{(a,b,1,0),(c,d,0,1)\} : a,b,c,d \in \mathbb{R}\}.$$

The cell $e(\sigma)$ is the wx-plane (in $wxyz$-space, i.e., $\mathbb{R}^4$). The cell $e(\tau)$ is all 2-dimensional subspaces of $\mathbb{R}^4$, except for the space when both $w = 0$ and $x = 0$. Thus, $e(\tau) = \mathrm{Gr}(2,4) \setminus e(\sigma)$.

It is easy to see from looking at canonical representatives that we can also write $e(\sigma)$ as

$$e(\sigma) = \{X \in Gr(k,n) : \dim(X \cap \mathbb{R}^{\sigma_i}) = i \ \text{ for } \ i = 1,\dots,k\}. \tag{2.1}$$

Proposition 19. *Let σ be a Schubert symbol for $\mathrm{Gr}(k,n)$. Then $e(\sigma)$ is a cell of dimension*

$$\dim(e(\sigma)) = \sum_{i=1}^{k} (\sigma_i - i), \quad \textit{with} \ \ 0 \le \dim(e(\sigma)) \le k(n-k).$$

Proof. Let $\mathbf{e}(\sigma)$ be the set of all canonical representatives of $k \times n$ matrices with basic columns $\sigma_1, \dots, \sigma_k$. Then there is a bijection $\mathbf{e}(\sigma) \to e(\sigma)$ given by $L \mapsto [L]$. The elements of $\mathbf{e}(\sigma)$ are exactly $k \times n$ matrices with σ_i-th column being the i-th standard basis vector and with 0s appearing to the right of the 1 in that column for $i = 1, \dots, k$. The other entries are arbitrary and uniquely determine the element of $\mathbf{e}(\sigma)$. We picture an arbitrary element of $\mathbf{e}(\sigma)$ as a matrix whose entries are 0, 1, or $*$, where $*$ represents an arbitrary real number, as in Example 17. The dimension of $e(\sigma)$ is the number of $*$ entries.

We count the number of $*$ entries by row. In the first row, there are $\sigma_1 - 1$, since the $*$ entries are exactly those preceeding the 1 in column σ_1. In the second row, a $*$ appears in each entry preceding the 1 in column σ_2, except for the entry in column σ_1, which is a 0. Thus, a $*$ appears $\sigma_2 - 2$ times in row 2. Similarly, $*$ appears $\sigma_i - i$ times in row i for all i. The total number of $*$s is $\sum_{i=1}^{k}(\sigma_i - i)$.

Consider Example 17, which has $\dim(e(2,4,7)) = 1 + 2 + 4 = 7$. For the leading 1 in the first row, there is only one $*$ preceding it, with 0s composing the rest of the row. This must be the case since the first basic column is in the second position. Similarly, for the leading 1 in the second row, there are two $*$s preceeding it, with 0s composing the rest of the row. Since the second basic column is in the fourth position, and the first column is already taking up one column to the left, there are only two open columns to the left of the second basic column. Using this same reasoning, we obtain 4 $*$s preceeding the 1 in the third basic column.

Reading off the $*$ entries left-to-right top-to-bottom then determines a homeomorphism $e(\sigma) \to \mathbb{R}^d$ where $d = \sum_{i=1}^{k}(\sigma_i - i) = \dim(e(\sigma))$. To show that $\mathbb{R}^d \cong B^d$ for any $d \in \mathbb{N}$, let the mapping ω be defined as

$$\begin{aligned} \omega : B^d &\to \mathbb{R}^d \\ x &\mapsto \frac{x}{1 - |x|} \end{aligned} \qquad\qquad \begin{aligned} \omega^{-1} : \mathbb{R}^d &\to B^d \\ x &\mapsto \frac{x}{1 + |x|}. \end{aligned}$$

Since ω is a rational function with a non-vanishing denominator, ω is continuous. The inverse ω^{-1} is similarly continuous. Thus, ω is a homeomorphism, so $\mathbb{R}^d \cong B^d$. Since $e(\sigma) \cong \mathbb{R}^d$, this also implies that $e(\sigma) \cong B^d$ as well. Thus, $e(\sigma)$ is a *d-cell* (Definition 8). $\square$

Example 20. Notice that the cell $e(n-k+1, n-k+2, \ldots, n)$ for $\mathrm{Gr}(k,n)$ has dimension $k(n-k)$:

$$\begin{aligned}\dim(e(n-k+1, n-k+2, \ldots, n)) &= \sum_{i=1}^{k}(\sigma_i - i)\\ &= (n-k+1-1) + (n-k+2-2) + \cdots + (n-k)\\ &= (n-k) + (n-k) + \cdots + (n-k) = k(n-k).\end{aligned}$$

This cell is an element in the standard atlas for $\mathrm{Gr}(k,n)$ described in Chapter 1. For instance, in $\mathrm{Gr}(3,9)$:

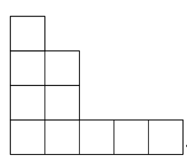
$$e(7,8,9) = \begin{bmatrix} * & * & * & * & * & * & 1 & 0 & 0\\ * & * & * & * & * & * & 0 & 1 & 0\\ * & * & * & * & * & * & 0 & 0 & 1 \end{bmatrix}.$$

2.2.1 Young diagrams

We will now introduce Young diagrams as a combinatorial tool for encoding the Schubert symbols of $\mathrm{Gr}(k,n)$.

Definition 21. Let λ be a sequence $(\lambda_1, \ldots, \lambda_k)$ of non-increasing non-negative integers, and let λ_i be the ith integer of λ. First, create a row of λ_1 boxes. Next, aligned to the left, create a row of λ_2 boxes on top of the row of λ_1 boxes. Repeat this process for all λ_i to obtain a diagram containing rows of boxes of non-increasing order. This arrangement of boxes determined by λ is called a *Young diagram.*

Example 22. $\lambda = (5,2,2,1)$ produces the following Young diagram,

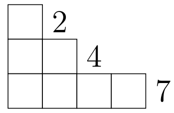

.

(Note that this is the French notation of a Young diagram. The standard English notation flips the diagram over the above x-axis. Given the shape of our canonical representatives, the French notation is more suitable here.)

Now we seek to define the correspondence between Schubert symbols of $\mathrm{Gr}(k,n)$ and Young diagrams. Consider Example 17 with $\mathrm{Gr}(3,9)$ and $\sigma = (2,4,7)$. We have shown that the cell $e(2,4,7)$ corresponds to the matrix

$$\begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0\\ * & 0 & * & 0 & * & * & 1 & 0 & 0 \end{bmatrix}.$$

Using this matrix, we can draw the cell's corresponding Young diagram by mimicking the shape of the $*$s as follows,



.

It is evident that $\lambda = (4, 2, 1)$. Additionally, the the numbers to the right of each column can be obtained by counting the edges touched by going down the diagram like stairs (touching the vertical edges as well) until touching the edge to the left of the number. These integers also denote the placement of the leading 1 in the corresponding row, giving us the Schubert symbol $\sigma = (2, 4, 7)$. In general, for a Schubert symbol σ of $\mathrm{Gr}(k, n)$, there is a bijection between λ and σ, given by

$$\lambda_{k+1-i} = \sigma_i - i, \quad \text{for} \quad i = 1, \ldots, k.$$

2.2.2 Schubert varieties

We now seek to define and discuss the Schubert varieties to help us develop the CW-decomposition of $\mathrm{Gr}(k, n)$.

Definition 23. For each Schubert symbol σ for $\mathrm{Gr}(k, n)$ define the *Schubert variety* Ω_σ to be the closure $\overline{e(\sigma)}$ of $e(\sigma)$ in $\mathrm{Gr}(k, n)$.

Proposition 24. *For each Schubert symbol σ,*

$$\Omega_\sigma = \bigcup_{\tau \leq \sigma} e(\tau) = \{X \in \mathrm{Gr}(k, n) : \dim(X \cap \mathbb{R}^{\sigma_i}) \geq i\}.$$

Proof. The second equality, $\bigcup_{\tau \leq \sigma} e(\tau) = \{X \in \mathrm{Gr}(k, n) : \dim(X \cap \mathbb{R}^{\sigma_i}) \geq i\}$, follows by looking at canonical forms of $\mathrm{Gr}(k, n)$ just as in Equation (2.1). We now concentrate on the first equality. Visualizing the canonical form of $e(\sigma)$, notice that all entries to the right of σ_i in row i are zeros. Therefore, considering the sub-determinants of $e(\tau)$, we have

$$\bigcup_{\tau \leq \sigma} e(\tau) = \{X \in \mathrm{Gr}(k, n) : \Delta_{j_1, \ldots, j_k}(X) = 0 \;\; \text{whenever} \;\; j_i > \sigma_i\}.$$

Each equation $\Delta_{j_1, \ldots, j_k} = 0$ defines a closed set. So $\bigcup_{\tau \leq \sigma} e(\tau)$ is the intersection of closed sets, hence, it is closed. Since $\bigcup_{\tau \leq \sigma} e(\tau)$ is closed and contains $e(\sigma)$,

$$\bigcup_{\tau \leq \sigma} e(\tau) \supseteq \overline{e(\sigma)} = \Omega_\sigma.$$

It remains to show that $\bigcup_{\tau \leq \sigma} e(\tau) \subseteq \overline{e(\sigma)}$. Let $\tau \leq \sigma$ and assume $\tau \neq \sigma$. Let $X \in e(\tau)$. We claim there is a curve $\gamma : \mathbb{R} \to \mathrm{Gr}(k, n)$ such that γ restricted to $\mathbb{R} \setminus \{0\}$ has image in $e(\sigma)$ and $\gamma(0) = X$. This shows that X is a limit point of $e(\sigma)$

and is, thus, an element of $\overline{e(\sigma)}$. To construct γ, let L be the canonical representation of X, and let t be an indeterminate. Define the matrix $\tilde{L}(t)$ by

$$\tilde{L}_{i,j}(t) = \begin{cases} t & \text{if } j = \sigma_i > \tau_i \\ L_{i,j} & \text{otherwise.} \end{cases}$$

For example, if $\tau = (1, 3, 5)$, $\sigma = (2, 4, 5)$, and

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 3 & 1 \end{bmatrix}.$$

Then,

$$\tilde{L}(t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 1 & t & 0 \\ 0 & 2 & 0 & 3 & 1 \end{bmatrix}.$$

Finally, let $\gamma : \mathbb{R} \to \mathrm{Gr}(k, n)$ where $\gamma(t) = [\tilde{L}(t)]$. So $[\tilde{L}(t)] \in e(\sigma)$ whenever $t \neq 0$, and when $t = 0$, we obtain some element $X = [\tilde{L}(0)] \in e(\tau)$. Thus, X is indeed a limit point of $e(\sigma)$, so $X \in \overline{e(\sigma)}$ and $\cup_{\tau \leq \sigma} e(\tau) \subseteq \overline{e(\sigma)}$. Since we have shown both directions of containment,

$$\Omega_\sigma = \overline{e(\sigma)} = \bigcup_{\tau \leq \sigma} e(\tau).$$

□

2.2.3 Characteristic maps

Proposition 25. *Let $V \in e(\sigma)$. There exists a unique orthonormal basis $u_1, \ldots, u_k$ for V such that $u_i \in V \cap \mathbb{R}^{\sigma_i}$ and $u_i \cdot e_{\sigma_i} > 0$ and $u_i \cdot e_j = 0$ for $j > \sigma_i$.*

Proof. To start, we know $\dim(V \cap \mathbb{R}^{\sigma_1}) = 1$. Say $V \cap \mathbb{R}^{\sigma_1} = \mathrm{Span}\{v\}$. If $u \in V \cap \mathbb{R}^{\sigma_1}$, then $u = \alpha v$ for some $\alpha \in \mathbb{R}$. But then $|u| = 1$ implies $\alpha = \pm\frac{1}{|v|}$. The condition $u \cdot e_{\sigma_i} > 0$ then specifies α. We proceed by induction supposing that $u_1, \ldots, u_{l-1}$ have been chosen for some $l \leq k$. Since $\dim(V \cap \mathbb{R}^{\sigma_{l-1}}) = \dim(V \cap \mathbb{R}^{\sigma_l})$, there exists $w \in V \cap \mathbb{R}^{\sigma_l}$ such that $V \cap \mathbb{R}^{\sigma_l} = \mathrm{Span}\{u_1, \ldots, u_{l-1}, w\}$. So if $u \in V \cap \mathbb{R}^{\sigma_l}$, there exists $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_{l-1} \in \mathbb{R}$ such that

$$u = \lambda w + \sum_{i=1}^{l-1} \lambda_i u_i.$$

Now we will proceed with the Gram-Schmidt process. For u to be orthogonal to each u_i, we need

$$0 = u \cdot u_j = \lambda w \cdot u_j + \sum_{i=1}^{l-1} \lambda w \cdot u_j + \lambda_j.$$

Hence, $\lambda_j = -\lambda w \cdot u_j$ for $j = 1, \ldots, l-1$. For $|u| = 1$, we need

$$\begin{aligned}
1 = u \cdot u &= (\lambda w + \sum_{i=1}^{l-1} \lambda_i u_i)(\lambda w + \sum_{i=1}^{l-1} \lambda_i u_i) \\
&= \lambda^2 w \cdot w + 2\lambda \sum_{i=1}^{l-1} \lambda_i (u_i w) + 2 \sum_{i \neq j} \lambda_i \lambda_j u_i \cdot u_j + \sum_{i=1}^{l-1} \lambda_i^2 u_i \cdot u_i \\
&= \lambda^2 w \cdot w - 2 \sum_{i=1}^{l-1} \lambda_i^2 + \sum_{i=1}^{l-1} \lambda_i^2 \\
&= \lambda^2 w \cdot w - \sum_{i=1}^{l-1} \lambda_i^2.
\end{aligned}$$

Thus,

$$\lambda = \pm \sqrt{1 + \sum_{i=1}^{l-1} \lambda_i^2}.$$

The requirement that $u_l \cdot e_{\sigma_l} > 0$ then determines a unique choice for λ. The result follows by induction. $\square$

Now, let D_σ be sequences of orthonormal vectors $(u_1, u_2, \ldots, u_k)$ such that, $u_i \in \mathbb{R}^{\sigma_i}$ and $u_i \cdot e_{\sigma_i} \geq 0$. Now we can define the map

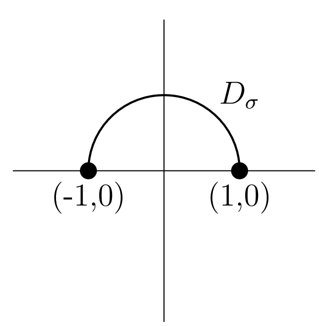
$$\begin{aligned}
s_\sigma : D_\sigma &\to \Omega_\sigma \\
(u_1, \ldots, u_k) &\mapsto \operatorname{span}\{u_1, \ldots, u_k\}.
\end{aligned} \tag{2.2}$$

Note that D_σ is a closed subset of $\mathbb{R}^{\sum \sigma_i}$. Denote the interior $\mathring{D}_\sigma$.

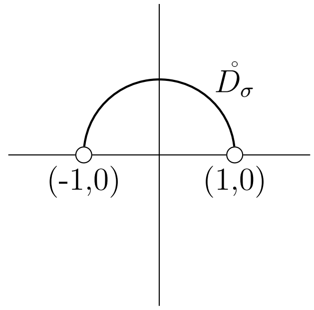
Example 26. Consider Gr$(2,3)$ and let $\sigma = (1,3)$. Then elements of D_σ have the form (u_1, u_2),

$$u_1 = (1,0,0) \qquad\qquad u_2 = (0, b, \sqrt{1-b^2}),$$

with $-1 \leq b \leq 1$. So D_σ is homeomorphic to the set of points $\{(b, \sqrt{1-b^2}) \in \mathbb{R}^2 : -1 \leq b \leq 1\}$. We can now visualize the closed set D_σ as a one-dimensional subset of $\mathbb{R}^2$:



The interior $\mathring{D}_\sigma$ is when $\sqrt{1-b^2} \neq 0$. Similarly, we can visualize the open set $\mathring{D}_\sigma$:



Notice that $\mathring{D}_\sigma$ is homeomorphic to $e(\sigma)$ for $\sigma = (1,3)$, since

$$e(1,3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & 1 \end{bmatrix} \;\middle|\; * \in \mathbb{R} \right\} \cong \mathbb{R} \cong \text{ any open interval of } \mathbb{R},$$

and $\mathring{D}_\sigma$ is homeomorphic to an open interval of $\mathbb{R}$. Now we seek to define and prove a more general case.

Proposition 27. *The map s_σ in Equation* (2.2) *restricts to a homeomorphism of the interior of D_σ with $e(\sigma)$.*

Proof. Let $\mathring{s}_\sigma$ be the restriction of s_σ to the interior of D_σ,

$$\mathring{D}_\sigma = \{(u_1, \ldots, u_k) \in D_\sigma : u_i \cdot e_{\sigma_i} > 0 \text{ for } i = 1, \ldots, k\}.$$

For $(u_1, \ldots, u_k) \in \mathring{D}_\sigma$, let $V = \mathrm{Span}\{u_1, \ldots, u_k\}$. It follows $\dim(V \cap \mathbb{R}^{\sigma_i}) = \dim(V \cap \mathbb{R}^{\sigma_i - 1}) + 1$ for all $i = 1, \ldots, k$. So $V \in e(\sigma)$. Thus, we have a well defined mapping

$$\mathring{s}_\sigma : \mathring{D}_\sigma \to e(\sigma).$$

By Proposition 25, the mapping is bijective. Next, we must show continuity for both directions. The forwards mapping sends $(u_1, \ldots, u_k)$ to the matrix with rows $u_1, \ldots, u_k$. Since $\mathrm{Gr}(k,n)$ has the quotient topology, the mapping $\mathring{s}_\sigma$ is continuous. Its inverse is continuous via the continuity of the Gram-Schmidt process. □

Proposition 28. *The image of ∂D_σ, the boundary of D_σ, lies in the* $\dim(\Omega_\sigma) - 1$ *skeleton of Ω_σ.*

Proof. Let $u = (u_1, \ldots, u_k) \in \partial D_\sigma$. Then there exists i such that $u_i \cdot e_{\sigma_i} = 0$. Hence, $s_\sigma(u) \notin e(\sigma)$. However, $s_\sigma(u) \in \Omega_\sigma = \cup_{\tau \leq \sigma} e(\tau)$. Therefore, $s_\sigma(u)$ is in $\cup_{\tau \lneq \sigma} e(\tau)$, the $\dim(\Omega_\sigma) - 1$ skeleton of Ω_σ. □

Definition 29. The *special orthogonal group*, $SO(n)$, consists of all orthogonal matrices of determinant 1. More explicitly,

$$SO(n) = \{n \times n \text{ matrices } M : MM^\top = I_d \text{ and } \det M = 1\}.$$

If $M \in SO(n)$, then there exists an orthogonal basis for $\mathbb{R}^n$ such that, with respect to this basis, M has the form

$$\begin{bmatrix} R_1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & R_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},$$

where each R_i is a 2×2 rotation matrix,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For reference, see Orthogonal group (2005).

Proposition 30. *The space D_σ is homeomorphic to the product*

$$D_0^{\sigma_1-1} \times D_0^{\sigma_2-2} \times \cdots \times D_0^{\sigma_k-k},$$

where each $D_0^{\sigma_i-i}$ is the set of vectors $v \in \mathbb{R}^{\sigma_i} \subseteq \mathbb{R}^n$ with the properties

$$\begin{aligned} |v| &= 1 \\ v \cdot e_{\sigma_i} &\geq 0 \\ v \cdot e_{\sigma_j} &= 0 \text{ for } j < i. \end{aligned}$$

Thus, D_σ is homeomorphic to the disk $D^{\dim(e(\sigma))}$ where $\dim(e(\sigma)) = \sum_{i=1}^{k}(\sigma_i - i)$.

Proof. First, note that the product of disks $D^{\sigma_1-1} \times \cdots \times D^{\sigma_k-k}$ is a compact convex set, and hence homeomorphic to $D^{\dim(e(\sigma))}$. Using the definition of $D_0^{\sigma_i-i}$ in Proposition 30, notice that $D_0^{\sigma_i-i}$ is composed of vectors of the following form:

$$v = (v_1, v_2, \ldots, v_{\sigma_i-1}, \sqrt{1 - v_1^2 - v_2^2 - \cdots - v_{\sigma_i}^2}, 0, 0, \ldots, 0),$$

such that $v \cdot e_{\sigma_j} = 0$ for $j < i$. Note that $D_0^{\sigma_i-i}$ is homeomorphic to D^{σ_i-i}. For example, let $\sigma = (3, 5, 6)$ and $n = 8$. Then, for instance, there is a homeomorphism

$$\begin{aligned} D^3 &\to D_0^{\sigma_3-3} \\ (a_1, a_2, a_3) &\mapsto (a_1, a_2, 0, a_3, 0, \sqrt{1 - a_1^2 - a_2^2 - a_3^2}, 0, 0). \end{aligned}$$

Using the definition of $v \in D_0^{\sigma_i-i}$, we can create such mappings for all $i = 1, \ldots, k$. Thus, so far we have shown that

$$D^{\dim(e(\sigma))} \cong D^{\sigma_1-1} \times \cdots \times D^{\sigma_k-k} \cong D_0^{\sigma_1-1} \times \cdots \times D_0^{\sigma_k-k}.$$

Next, to show that D_σ and $D^{\dim(e(\sigma))}$ are homeomorphic, we seek to construct a homeomorphism between D_σ and $D_0^{\sigma_1-1} \times \cdots \times D_0^{\sigma_k-k}$. First, we construct a homeomorphism

$$f_\sigma : D_\sigma \to D_0^{\sigma_1-1} \times D_{\sigma'},$$

where $\sigma' = (\sigma_2-1, \sigma_3-1, \ldots, \sigma_k-1) = (\sigma'_1, \sigma'_2, \ldots, \sigma'_{k-1})$ and $D_{\sigma'}$ is associated with the cell $e(\sigma')$ in $\mathrm{Gr}(k-1, n-1)$. The result then follows by induction on k. Let $T_v \in SO(n)$, which rotates v to e_{σ_1} in the plane spanned by v and e_{σ_1}, and is the identity in the orthogonal complement of this plane. As a matrix, it will take the form of a rotation sub-matrix in the upper left corner to rotate v to e_{σ_1}, with the rest of the diagonal as the identity matrix. Let $\mathbb{R}^n_{\sigma_1} = \{x \in \mathbb{R}^n : x_{\sigma_1} = 0\}$ and define the isomorphism $p : \mathbb{R}^n_{\sigma_1} \to \mathbb{R}^{n-1}$ by dropping the σ_1-th component. Now we can define f_σ as

$$f_\sigma(v_1, \ldots, v_k) = (v_1, p(T_{v_1}v_2), p(T_{v_1}v_3), \ldots, p(T_{v_1}v_k)).$$

Notice that v_1 takes the form

$$v_1 = (b_1, b_2, \ldots, b_{\sigma_1-1}, \sqrt{1-b_1^2-b_2^2-\cdots-b_{\sigma_1-1}^2}, 0, \ldots, 0),$$

so $v_1 \in D_0^{\sigma_1-1}$. Thus, v_1 is mapped to itself, and $(v_2, \cdots, v_k)$ is mapped to an element in $D_{\sigma'}$. However, we must still show that $(p(T_{v_1}v_2), p(T_{v_1}v_3), \ldots, p(T_{v_1}v_k))$ is indeed in $D_{\sigma'}$. First, if $i > 1$, then

$$0 = v_1 \cdot v_i = T_{v_1}v_1 \cdot T_{v_1}v_i = e_{\sigma_1} \cdot T_{v_1}v_i.$$

So the σ_1-th component of each $T_{v_1}v_i$ is 0 for $i > 1$. Next we show

$$T_{v_1}v_2, \ldots, T_{v_1}v_k \text{ are orthonormal,} \tag{2.3}$$
$$e_{\sigma_i} \cdot T_{v_1}v_i \geq 0 \text{ for } i > 1, \tag{2.4}$$
$$e_j \cdot T_{v_1}v_i = 0 \text{ if } j > \sigma_i. \tag{2.5}$$

Let $\delta(i,j)$ be 1 if $i = j$ and 0 if $i \neq j$. Then, using the fact that rotations do not change length nor angles between vectors, we have for $i, j > 1$,

$$T_{v_1}v_i \cdot T_{v_1}v_j = v_i \cdot v_j = \delta(i,j).$$

This proves (2.3). For (2.4), we use the fact that $T_{v_1}e_{\sigma_i} = e_{\sigma_1}$ for $i > 1$:

$$e_{\sigma_i} \cdot T_{v_1}v_i = T_{v_1}e_{\sigma_i} \cdot T_{v_1}v_i = e_{\sigma_i} \cdot v_i \geq 0.$$

Equation (2.5) follows similarly. Since the σ_1-th component of each $T_{v_1}v_i$ for $i > 1$ is 0, each of (2.3), (2.4), and (2.5) hold after applying p:

$$\begin{aligned}
&p(T_{v_1}v_2), \ldots, p(T_{v_1}v_k) \text{ are orthonormal,}\\
&e_{\sigma'_i} \cdot p(T_{v_1}v_i) = e_{\sigma_i-1} \cdot p(T_{v_1}v_i) = pe_{\sigma_i} \cdot p(T_{v_1}v_i) = 0 \text{ for } i > 1,\\
&e_{j-1} \cdot p(T_{v_1}v_i) = pe_j \cdot p(T_{v_1}v_i) = 0 \text{ if } j > \sigma_i, \text{ i.e., } j-1 > \sigma'_i.
\end{aligned}$$

Thus, we obtain the desired $(p(T_{v_1}v_2), p(T_{v_1}v_3), \ldots, p(T_{v_1}v_k)) \in D_{\sigma'}$. Lastly, the inverse homeomorphism of f_σ is

$$f_\sigma^{-1}(v_1, u_2, \cdots, u_k) = (v_1, T_{v_1}^{-1}(p^{-1}u_2), \ldots, T_{v_1}^{-1}(p^{-1}u_k)),$$

where p^{-1} adds a 0 in the σ_1-th component of a vector in $\mathbb{R}^{n-1}$ and $T_{v_1}^{-1}$ is the inverse matrix of $T_{v_1} \in SO(n)$. So D_σ is indeed homeomorphic to $D_0^{\sigma_1} \times \cdots \times D_0^{\sigma_k - k}$, which implies that $D_\sigma \cong D^{\dim(e(\sigma))}$. □

Theorem 31. *The set of $e(\sigma)$ as σ ranges over Schubert symbols for* $\mathrm{Gr}(k,n)$ *forms a CW decomposition.*

Proof. By Proposition 19, each $e(\sigma)$ is indeed a cell. Each $e(\sigma)$ is uniquely determined by its canonical representative. Hence, $\mathrm{Gr}(k,n)$ is a disjoint union of the $e(\sigma)$,

$$\mathrm{Gr}(k,n) = \bigsqcup_\sigma e(\sigma).$$

Since there are $\binom{n}{k}$ Schubert symbols for a given $\mathrm{Gr}(k,n)$, there are finitely many $e(\sigma)$ cells in its CW decomposition. For a finite CW complex, it only remains to show there are characteristic maps

$$\Phi_{e(\sigma)} : D^{\dim(e(\sigma))} \to \Omega_\sigma$$

for each Schubert symbol σ, such that (i) restricting $\Phi_{e(\sigma)}$ to the interior of $D^{\dim(e(\sigma))}$ gives a homeomorphism

$$\mathring{\Phi}_{e(\sigma)} : \mathring{D}^{\dim(e(\sigma))} \to e(\sigma),$$

and (ii) the image of $\Phi_{e(\sigma)}$ restricted to the boundary $\partial D^{\dim(e(\sigma))}$ is contained in the $(\dim(e(\sigma)) - 1)$-skeleton of the CW complex

$$\Phi_{e(\sigma)}(\partial D^{\dim(e(\sigma))}) \subseteq \bigcup_{\tau \lneq \sigma} e(\tau).$$

First, fix a homeomorphism $F_\sigma : D^{\dim(e(\sigma))} \to D_\sigma$. We have shown that such an F_σ exists in the proof of Proposition 30. Next, recall the mapping $s_\sigma : D_\sigma \to \Omega_\sigma$. Now we can define the characteristic mapping

$$\Phi_{e(\sigma)} = s_\sigma \circ F_\sigma : D^{\dim(e(\sigma))} \xrightarrow{F_\sigma} D_\sigma \xrightarrow{s_\sigma} \Omega_\sigma.$$

By Proposition 27, we have already shown that $\mathring{s}_\sigma : \mathring{D}_\sigma \to e(\sigma)$ is a homeomorphism. Using that and the fact that homeomorphisms map interiors to interiors, so we obtain

$$\mathring{\Phi}_{e(\sigma)} = \mathring{s}_\sigma \circ \mathring{F}_\sigma : \mathring{D}^{\dim(e(\sigma))} \xrightarrow{\mathring{F}_\sigma} \mathring{D}_\sigma \xrightarrow{\mathring{s}_\sigma} e(\sigma).$$

A composition of homeomorphisms is a homeomorphism, condition (i) is satisfied. The proof of condition (ii) follows similarly. Proposition 28 shows that $s_\sigma(\partial D_\sigma) \subseteq \bigcup_{\tau \lneq \sigma} e(\tau)$. Since homeomorphisms map boundaries to boundaries, we have

$$\Phi_{e(\sigma)}(\partial D^{\dim(e(\sigma))}) = s_\sigma(F_\sigma(\partial D^{\dim(e(\sigma))})) = s_\sigma(\partial D_\sigma) \subseteq \bigcup_{\tau \lneq \sigma} e(\tau).$$

□

Lastly, we present an example that shows that our CW decomposition of $\mathrm{Gr}(k, n)$ is not regular: the characteristic maps are not necessarily homeomorphisms.

Example 32. $\mathrm{Gr}(1, 2)$ has two Schubert symbols, $\sigma = (2)$ and $\tau = (1)$. The cell $e(\sigma)$ is the set of points in $\mathrm{Gr}(1, 2)$ whose canonical representatives are 1×2 matrices of the form $\begin{bmatrix} a & 1 \end{bmatrix}$ with $a \in \mathbb{R}$. By scaling, we can also take unique representatives of the form $\begin{bmatrix} a & \sqrt{1-a^2} \end{bmatrix}$ such that $a \in (-1, 1)$. The cell $e(\tau)$ is the single point whose canonical representative is $\begin{bmatrix} 1 & 0 \end{bmatrix}$. In $\mathbb{R}^2$, $e(\sigma)$ is the set of non-horizontal lines through the origin, and $e(\tau)$ is the single horizontal line through the origin. We have

$$D_\sigma = \{(a, \sqrt{1-a^2}) : a \in [-1, 1]\},$$

a closed semi-circle in $\mathbb{R}^2$ (see Example 26 for visualization), and

$$\begin{aligned} s_\sigma : D_\sigma &\to \overline{e(\sigma)} \\ (a, \sqrt{1-a^2}) &\mapsto \left[\begin{bmatrix} a & \sqrt{1-a^2} \end{bmatrix} \right]. \end{aligned}$$

(The square brackets around the matrix are used to denote its equivalence class.) There are two points in the boundary of D_σ:

$$\partial D_\sigma = \{(a, \sqrt{1-a^2}) : a = \pm 1\} = \{(1, 0), (-1, 0)\}.$$

Therefore,

$$s_\sigma(D_\sigma) = \left\{ \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \right], \left[\begin{bmatrix} -1 & 0 \end{bmatrix} \right] \right\} = \left\{ \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \right] \right\} = e(\tau),$$

since $\begin{bmatrix} -1 & 0 \end{bmatrix}$ is in the same equivalence class as $\begin{bmatrix} 1 & 0 \end{bmatrix}$, and both represent the horizontal line in $\mathbb{R}^2$. Since the mapping s_σ sends two points in D_σ to a single point in $\overline{e(\sigma)}$, it is not injective. Hence, s_σ is not a homeomorphism.

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